SUBMISSION TO PRODUCTIVITY COMMISSION

David Bell[*], Adam Shao[†] and Estelle Liu[‡]

Mine Wealth + Wellbeing

9th September, 2016

1 Introduction

Despite the Superannuation Guarantee having existed for nearly 25 years a strong argument could be made that the super industry remains in an immature state regarding its ability to efficiently deliver strong and sustainable retirement incomes.

The industry has historically focused on accumulation and account balances, a peer group performance focus pervades, and there exists dispersion in the models and quality of fund governance.

The superannuation industry has only recently begun to consider the challenge of delivering sustainable income streams in retirement. Indeed in our estimation there are maybe just four super funds in Australia who have internal frameworks that stochastically model retirement outcomes which account for both investment and mortality risk.

We believe that there is a substantial efficiency gain from improving the design of post-retirement defaults. Solutions which better manage longevity risk, and a more informed trade-off between the level of expected retirement income and its certainty, are just two areas of substantial opportunity. Yet default solutions which better meet the retirement income challenge are just a first step towards delivering the greater efficiency that the Productivity Commission, and policymakers in general, seek.

As one example we believe there is significant improvement in efficiency by delivering personalised solutions which account for characteristics of the individual. Such an innovation would come at a cost and may require supportive regulatory changes from policymakers. Our prior expectation is that the benefits far exceed the costs. A second example is the consideration of household, rather than individual, outcomes. Appendix 1 includes an updated version of our original submission to Treasury on the Objective of Superannuation which addresses this issue. This example is provided simply to detail an area of allocative inefficiency that presently exists.

We commend the Productivity Commission's initial work in identifying a broad range of measurements which provide some insight into system efficiency. We acknowledge the efforts to identify measurements which provide ex-ante insight. However we believe that it remains difficult to identify the benefits of some of the forthcoming innovations that industry and policymakers will undoubtedly consider.

To this extent we have been leading an industry project which we believe has substantial application to both industry and policymakers. This project is known as the Trustee's Utility Function for Default Fund Design version 1 ("TUFv1"). We provide a brief background on the concept of a utility function and the project and then consider how the TUFv1 could be used to address a range of industry and policy questions.

A utility function is a mathematical representation of preferences. In finance terms we prefer higher returns than lower returns; we also tend to feel the pain of loss greater than the joy of an identically sized gain. A utility function can be developed to address many situations. In the context of retirement outcomes it becomes more complex: we need to consider a series of cash flows and consider whether the residual account value at death has any value. To address this we (about one year ago) created a working group of academics and industry experts to develop a representative utility function which can be used as a recognised starting point for industry when addressing fund design issues. The strong referencing to academic literature and robust debate amongst working group members provides assurance that this is a considered publication. TUFv1 will provide a well-considered, defensible starting point for Trustees in setting the objectives of their fund. However TUFv1 also represents a "straw man": funds would be encouraged to step away and create their own utility function if they believe they have better insight into the preferences of their members. It is intended that the TUFv1 will be broadly shared across industry and to also be published academically in an attempt to provide further assurance and a more consistent base for Australian superannuation research within academia. TUFv1 is nearing release (anticipated release is November 2016). Appendix 2 includes the most recent draft document. While the document may appear highly complex, this is necessary at the design stage; once agreed it is much easier to understand and use in practice.

Detailing some examples of the ways in which the TUFv1 can be applied may illustrate to Productivity Commission members its broad potential application in objectively addressing many complex issues:

1. Placing a value on the benefits of low volatility, thereby balancing the focus on higher outcomes
2. Determination of post-retirement fund design
3. Use by ratings groups to assess superannuation funds
4. An input into efficient prioritisation of projects by super funds
5. Estimating the benefits of incorporating personalisation

6. Assessing the opportunity cost of managing portfolios for peer group risk
7. Assessing the system-wide welfare benefits of regulatory change to enable product availability (an example is illustrated in Appendix 2)
8. Assessing the value of digital financial advice

We believe that there is merit in the Productivity Commission having, where possible, a common assessment framework for exploring complex questions which affect the efficient delivery of retirement outcomes (we envisage that there is a large range of projects that the Productivity Commission would like to explore). The TUFv1 may have relevant application in this respect as it provides an objective consistent starting point which is well considered. We welcome any engagement with Productivity Commission members on the issues raised in this submission. We have undertaken significant internal education at a Board and Executive level on the concept and use of utility functions and their application, which might be beneficial.

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Appendix 1

SUBMISSION TO OBJECTIVE OF SUPERANNUATION

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This paper is an updated version of the Submission to the Treasury based on the comments we received from presenting this paper at two conferences. In this version we focus on the importance of modelling household income instead of individual income in retirement.

1 Executive Summary

We recommend an alteration to the following:

"*To provide household income in retirement to substitute or supplement the Age Pension*".

Our case for this alteration is based on two modelled examples which illustrate how household assets may be inefficiently spent if a two person household is not recognised in combination. Any inefficiencies have a cost to households but also represent a cost to society as the age pension will not be used efficiently.

We note that while the institutional superannuation industry is not currently well placed from a technology perspective to manage for household outcomes it remains a relevant aspiration for the industry.

An additional extension to the objective of superannuation could be to consider whether superannuation has a role to play in providing a household capital safety net. This would support large expenditure health costs and aged care entry but there are residual concerns around the potential for bequests funded by tax-advantaged savings.

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A final reflection in our paper is the strong integration of Age Pension and superannuation. Any improvements in the ability to acknowledge this interrelation through coordinated retirement income policy development would have significant benefit from an outcomes and an efficiency perspective. If Treasury would like access to our modelling (MATLAB) we are happy to assist.

2 The Case for Recognising the Household

Households may consist of one individual but they may often consist of a couple. If this couple was provided quality financial advice they would leave with a coordinated plan for how they should spend down their respective superannuation accounts. By comparison, the institutional superannuation industry (by "institutional" we mean the superannuation industry ex of SMSF's) recognises all individuals as simply that: individuals.

Consumption plans for a two person household, established on the basis of the two being individuals, will be less efficiently devised. This will lead to less than optimal outcomes for the household which has a cost to society; in addition there is the cost of inefficient use of the Age Pension. We illustrate these inefficiencies through two worked examples. In the first example we consider a couple, each member beginning retirement with the same amount of retirement savings. This example allows us to introduce the concept of joint life expectancy, something which is not considered in individual financial plans. In the second example we consider the more practical situation where the balances of the household's two individuals are different at retirement. Both examples reveal significant opportunity costs.

Modelling retirement outcomes is a complex exercise. At Mine Wealth and Wellbeing we use stochastic models which consider investment and mortality risk. However for this exercise we use deterministic modelling whereby we assume that mortality events occur at their expected value. This is sufficient to illustrate the issue we have identified. The retirement income model is developed using assumptions based on the ASIC guide for generation of statement projection of members' retirement income. We make a simplifying assumption that a household targets a constant level of income whether one or two of the members are alive.

The main area of complexity introduced in these case studies is the estimation of joint life expectancy. A detailed note on this is provided in Appendix A.

3 Case Study 1

We assume the following scenario for a two member household:

- Member 1, female, age 65, superannuation balance $500k
- Member 2, male, age 65, superannuation balance $500k

We assume the two members are provided individual guidance by their respective superannuation fund, using a retirement tool similar to the ASIC model. The ASIC model assumes that the individual is part of a couple and assumes the individual's partner has the same retirement balance, an accurate assumption in this example. The recommended spending pattern for each member is shown in Figure 1 and Figure 2, respectively.

If we add these profiles together, maintaining our deterministic approach, we can see that the household does not plan to experience constant income: an income shock is expected to be

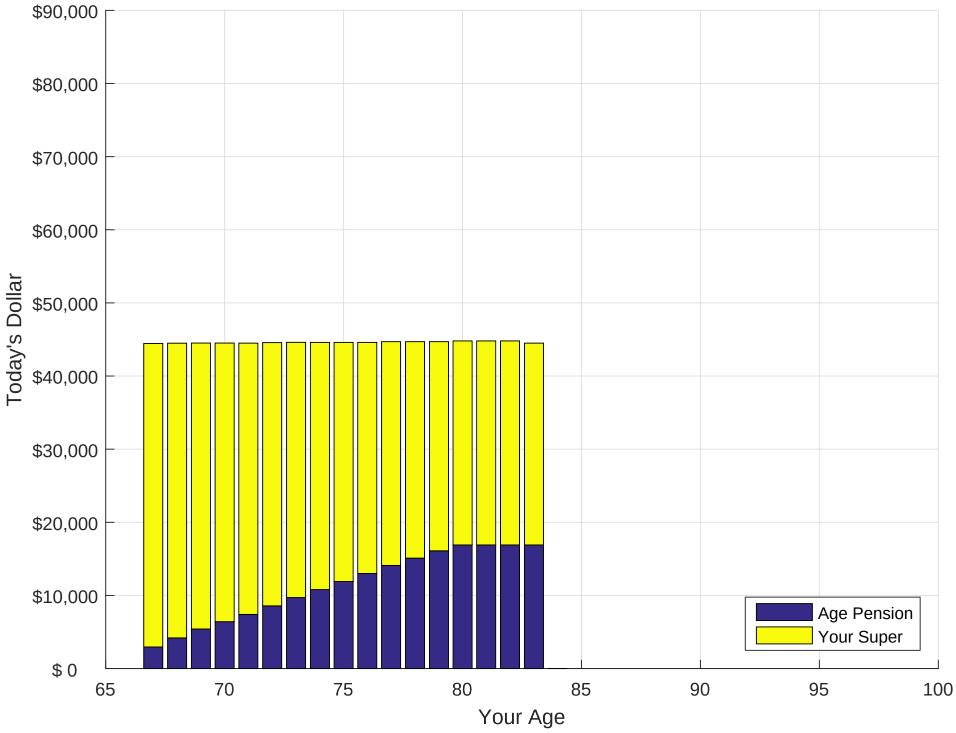

Figure 1: Spending pattern for a male aged 65 with a superannuation balance of $500k.

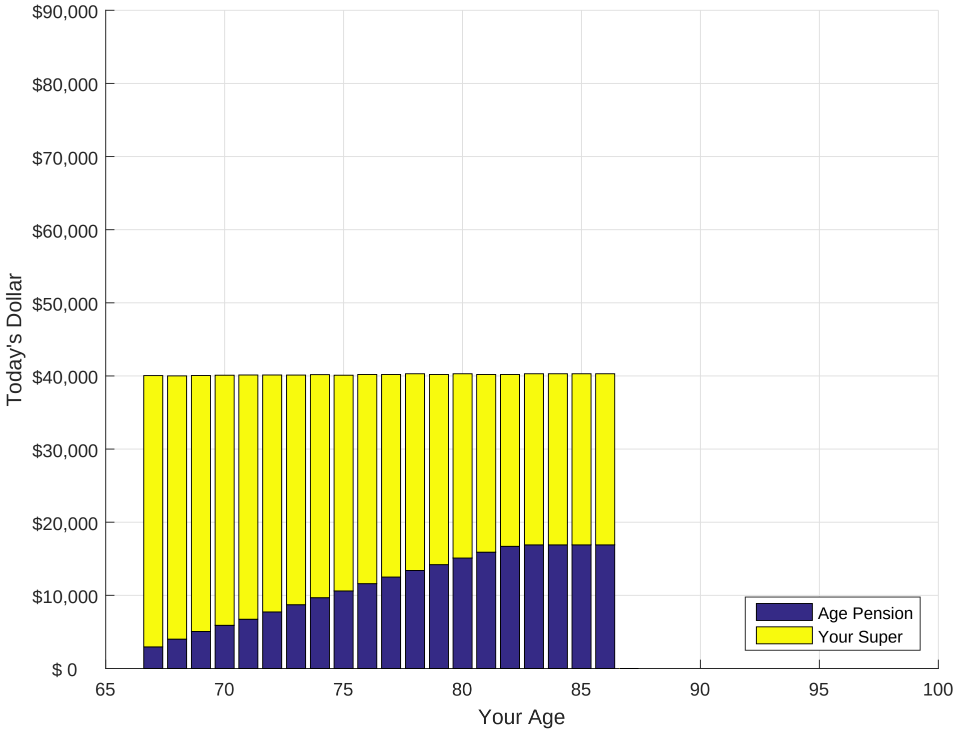

Figure 2: Spending pattern for a female aged 65 with a superannuation balance of $500k.

experienced when the first member of the couple passes away. The results are shown in Figure 3. (Note that this shortfall is marginally overstated because the single age pension payment received in reality is slightly higher than half the couple payment factored into our modelling).

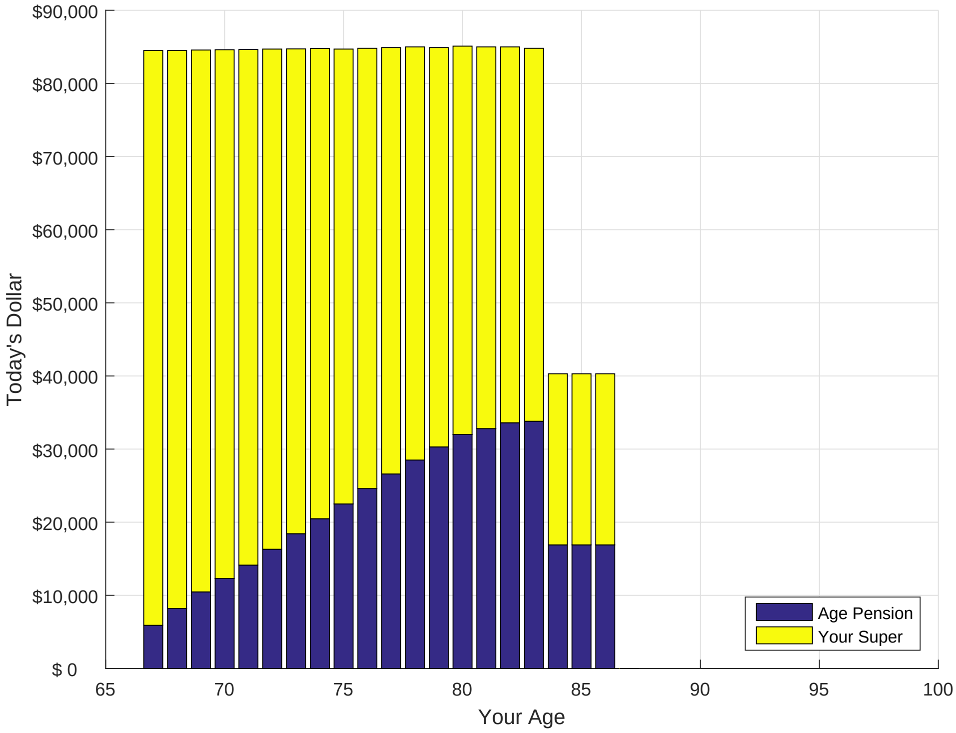

Figure 3: Combined spending pattern for a male and a female both aged 65, each with a superannuation balance of $500k.

Let us now consider the combined household model. We acknowledge that one member is likely to die younger than their individual life expectancy and one slightly older. With this knowledge they plan accordingly and we can see in Figure 4 and 5.

If we assume household spending changes minimally when only one member is alive, we can see in Figure 4 the household experience is constant. By comparing Figure 4 against Figure 3 we can see that due to individuals being provided individual advice the household collectively overspends and does not allow for the joint life expectancy. This represents inefficient use of superannuation savings and will also mean an inefficient use of the age pension.

If we assume household spending cuts by 30% when only one is alive (consistent with ASFA living standard), we can see in Figure 5 the household experience changes. By comparing Figure 5 against Figure 3 we can see that ignoring joint life expectancy results in overspending and the findings are consistent with the results you see in Figure 4.

4 Case Study 2

We now assume that balances at retirement are not the same for our two members of the household:

- Member 1, female, age 65, superannuation balance $250k
- Member 2, male, age 65, superannuation balance $750k

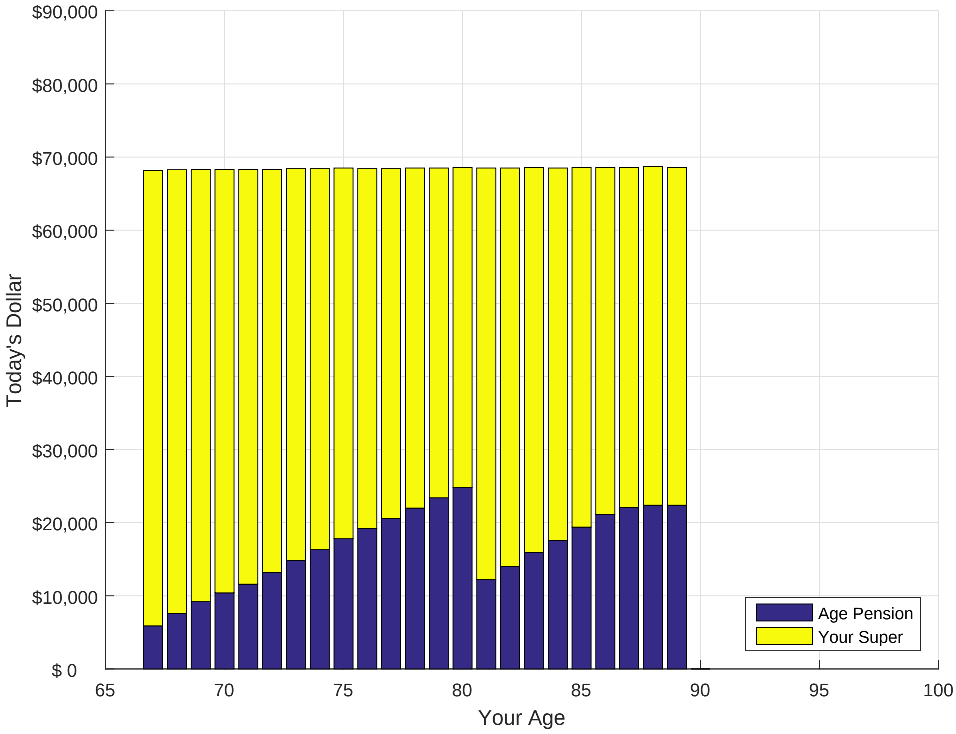

Figure 4: Combined household of a male and a female both aged 65 with a superannuation balance of $1m and assume no household spending cuts when only one is alive.

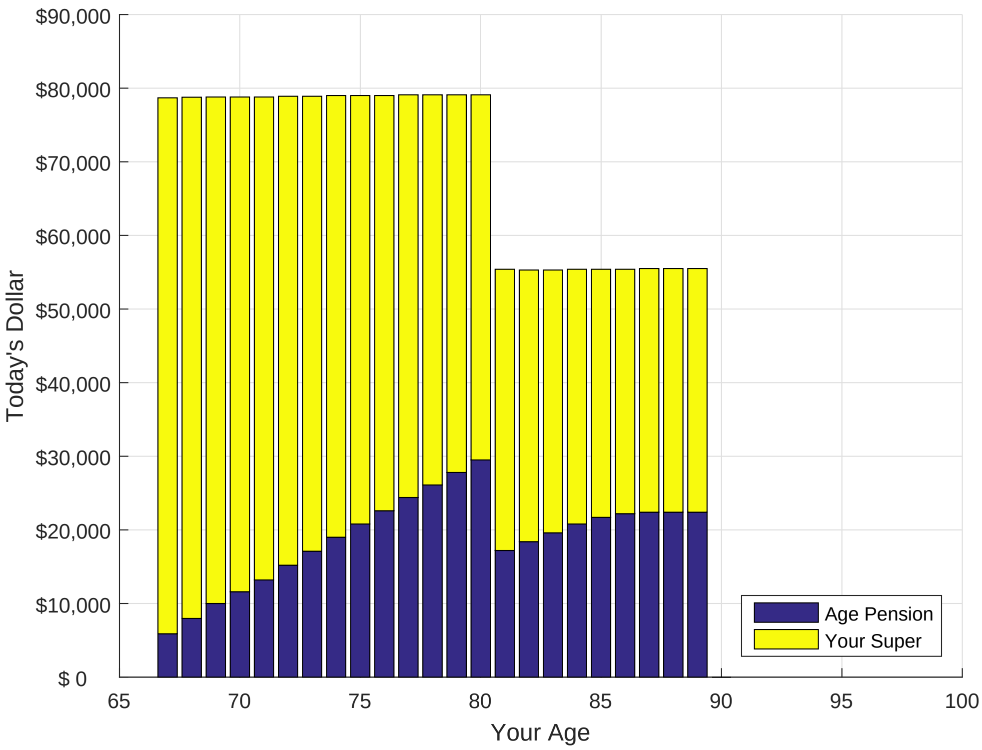

Figure 5: Combined household of a male and a female both aged 65 with a superannuation balance of $1m and assume household spending cuts by 30% when only one is alive.

We again assume the two members are provided individual guidance by their respective superannuation fund, using an ASIC-style calculator which assumes, incorrectly in this case, that the partner's superannuation balance is the same as their own. The recommended spending pattern for each member is shown in Figure 6 and Figure 7, respectively.

The combination of the two individual plans is shown in Figure 8.

In this case the drop in expected household income is much greater than in our previous example. If we allow for the joint life expectancy our profile is shown in Figure 9.

We can see by comparing Figure 9 against Figure 8 that a household based on two individual financial plans spends their savings much too quickly leaving the expectation of a significant income shock.

5 Comment on the Aspiration of Household Outcomes

The institutional superannuation industry is not currently well placed from a technology perspective to manage for household outcomes. Members of a household are often in different super funds. Indeed the superannuation industry has yet to make significant strides in terms of individual personalisation of member account design.

In our view this does not mean that the household should not be the focus of the superannuation objective. Indeed we believe that an objective can have an aspirational element to it. If the focus of the superannuation is the household then imagine the attention that will be directed by the superannuation industry on the technical and practical issues of maximising outcomes for households.

6 Should Superannuation Be a Source of Precautionary Savings?

An additional extension to the objective of superannuation could be to consider whether superannuation has a role to play in providing a household capital safety net. This is a vexing issue. On one hand compulsory superannuation represents forced savings which could otherwise be used to fund the creation of a precautionary savings pool which could be used for large expenditure items such as health costs and aged care entry. On the other hand superannuation is strongly tax advantaged (for mid to high income earners) and it could potentially be used as a bequest account.

We have no firm view on this issue.

7 Conclusion

We believe there is much benefit to gain from focusing on household, rather than individual, outcomes. The benefits are to the individual (more efficient use of their capital) and society (through more effective age pension payments). While industry may not yet be ready to focus on the couple, the word "household" could be a healthy aspirational nudge, directing industry's strategic focus in the right direction.

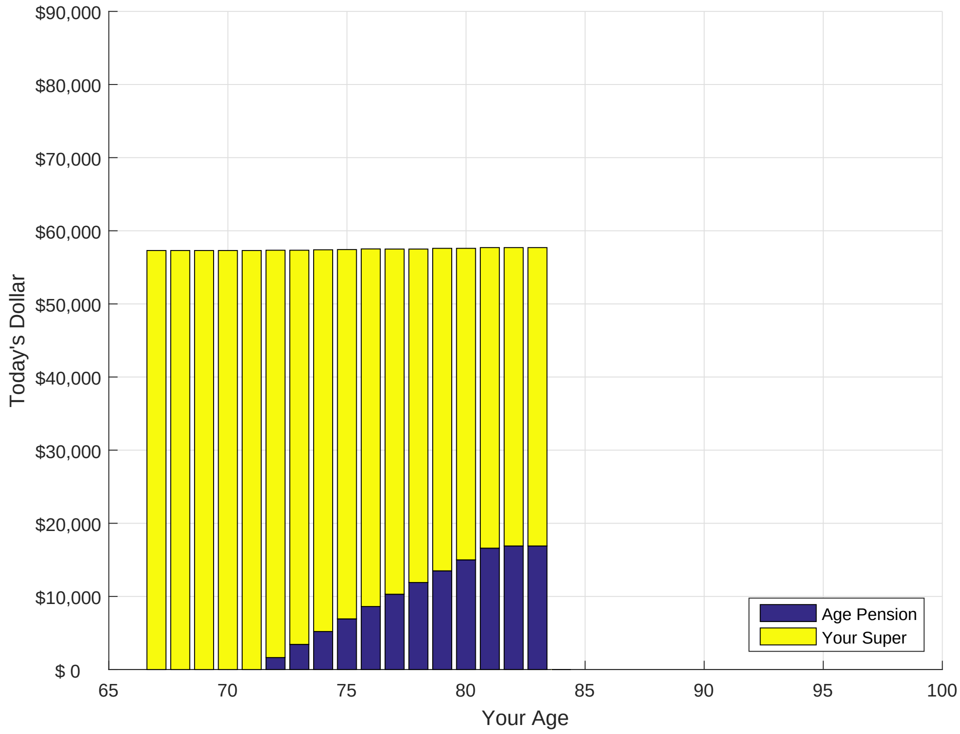

Figure 6: Spending pattern for a male aged 65 with a superannuation balance of $750k.

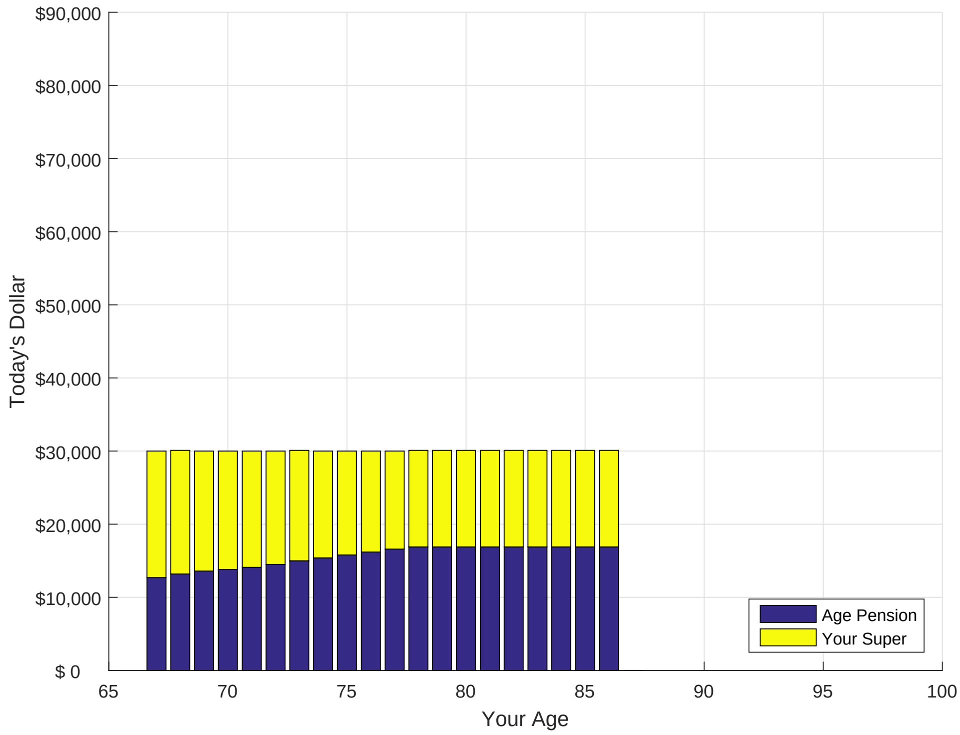

Figure 7: Spending pattern for a female aged 65 with a superannuation balance of $250k.

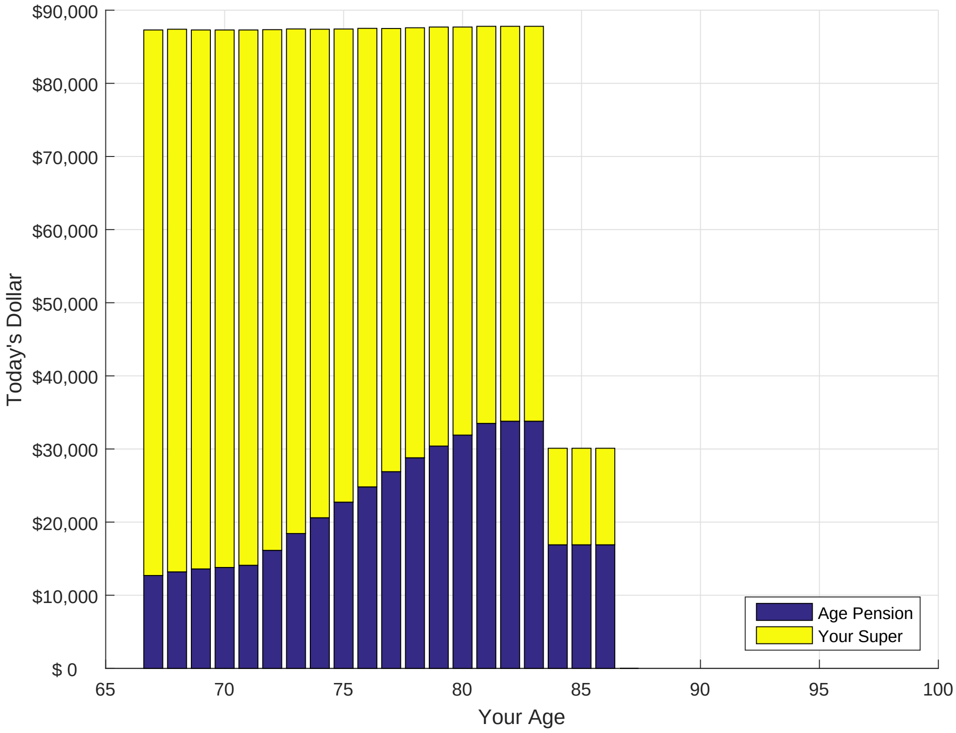

Figure 8: Combined spending pattern for a male and a female both aged 65, with a superannuation balance of $750k and $250k respectively.

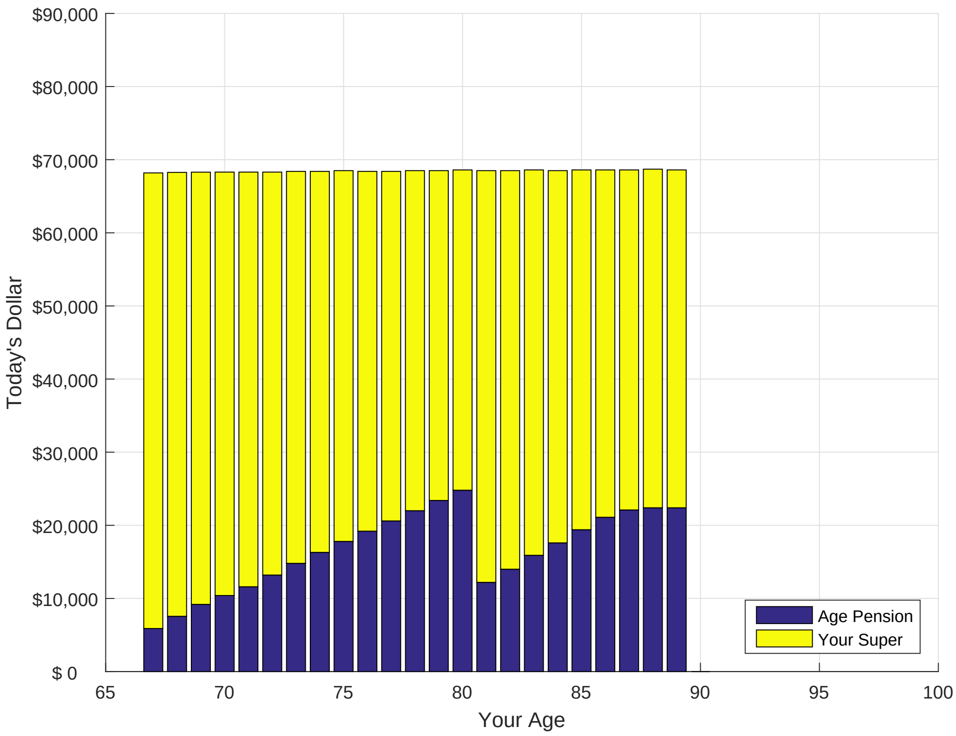

Figure 9: Combined household of a male and a female both aged 65 with a total superannuation balance of $1m and assume no household spending cuts when only one is alive. Note that in a combined household scenario the proportion of each member's superannuation balance does not make a difference so this figure is the same as Figure 4. For the convenience of comparison, we show it here again.

We are vexed on the issue of superannuation as a source of precautionary savings; it is a complex issue.

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Appendix A: Note on Calculating Residual Life Expectancy

We show our methods in calculating expected remaining lifetimes of each individual in a household, of the first death, and of the second death, respectively.

We consider households of a male and a female. Let T_1 and T_2 denote the random remaining lifetimes of the male and the female, respectively. The lifespan of the first death is the minimum lifespan of the two in a household, i.e. $\min\{T_1,\ T_2\}$; the lifespan of the second is the maximum lifespan of the two, i.e. $\max\{T_1,\ T_2\}$.

The residual life expectancy of the male and female is respectively calculated as the expected value of T_1 and T_2. The residual joint life expectancy, which means the expected remaining lifetime of the first death, is calculated as the expected value of $\min\{T_1,\ T_2\}$. The residual last survival life expectancy, which means the expected remaining lifetime of the second death, is calculated as the expected value of $\max\{T_1,\ T_2\}$.

Residual Life Expectancy of Individuals

For a male aged x, the residual life expectancy is calculated as follows:

$$\begin{aligned} \mathring{e}_{x,M} &= \mathbf{E}\Big(T_1\Big) \\ &= \sum_{t=1}^{\omega-x} {}_tp_{x,M} + \frac{1}{2}, \end{aligned} \tag{1}$$

where ω denotes the maximum attainable age, and ${}_tp_{x,M}$ denotes the probability that an x-year-old male survives to age $x+t$. We add half a year on the right hand side of the above equation based on the assumption that death times within integer ages follow a uniform distribution.

Similarly, the residual life expectancy for a female aged y is calculated as:

$$\begin{aligned} \mathring{e}_{y,F} &= \mathbf{E}\Big(T_2\Big) \\ &= \sum_{t=1}^{\omega-y} {}_tp_{y,F} + \frac{1}{2}, \end{aligned} \tag{2}$$

where ${}_tp_{y,M}$ denotes the probability that a y-year-old female survives to age $y+t$.

Residual Joint Life Expectancy

The residual joint life expectancy is calculated as follows:

$$\begin{aligned} \mathring{e}_{xy} &= \mathbf{E}\Big(\min\{T_1,\ T_2\}\Big) \\ &= \sum_{t=1}^{\omega-\min\{x,y\}} {}_tp_{xy} + \frac{1}{2}, \end{aligned} \tag{3}$$

where ${}_tp_{xy}$ denotes the joint life survival probability (that both lives survive to the next t years).

We assume the mortality of two lives of a couple are independent, so the joint life survival probability is calculated as the product of each individual's survival probability, i.e.

$${}_tp_{xy} = {}_tp_{x,M}\ {}_tp_{y,F}. \tag{4}$$

Residual Last Survivor Life Expectancy

We have the following equation on the random lifetimes:

$$\min\{T_1,\ T_2\} + \max\{T_1,\ T_2\} \quad = \quad T_1 + T_2. \tag{5}$$

The expected values of both sides in the above equation are equal, i.e.:

$$\mathbf{E}\Big(\min\{T_1,\ T_2\}\Big) + \mathbf{E}\Big(\max\{T_1,\ T_2\}\Big) \quad = \quad \mathbf{E}\Big(T_1\Big) + \mathbf{E}\Big(T_2\Big). \tag{6}$$

Let $\mathring{e}_{\overline{xy}}$ denote the residual last survivor life expectancy, so we have

$$\mathring{e}_{xy} + \mathring{e}_{\overline{xy}} \quad = \quad \mathring{e}_{x,M} + \mathring{e}_{y,F}. \tag{7}$$

Therefore, we calculate the residual last survivor life expectancy as follows:

$$\mathring{e}_{\overline{xy}} \quad = \quad \mathring{e}_{x,M} + \mathring{e}_{y,F} - \mathring{e}_{xy}. \tag{8}$$

Illustrations

Using the Australian 2009 period life table and based on the assumption that two lives of a couple have independent mortality, we calculate residual life expectancies for joint lives. Figure 11 shows residual life expectancies for individuals and for joint lives of the same age. Figure 12 shows the residual last survivor life expectancy for joint lives of different combinations of ages.

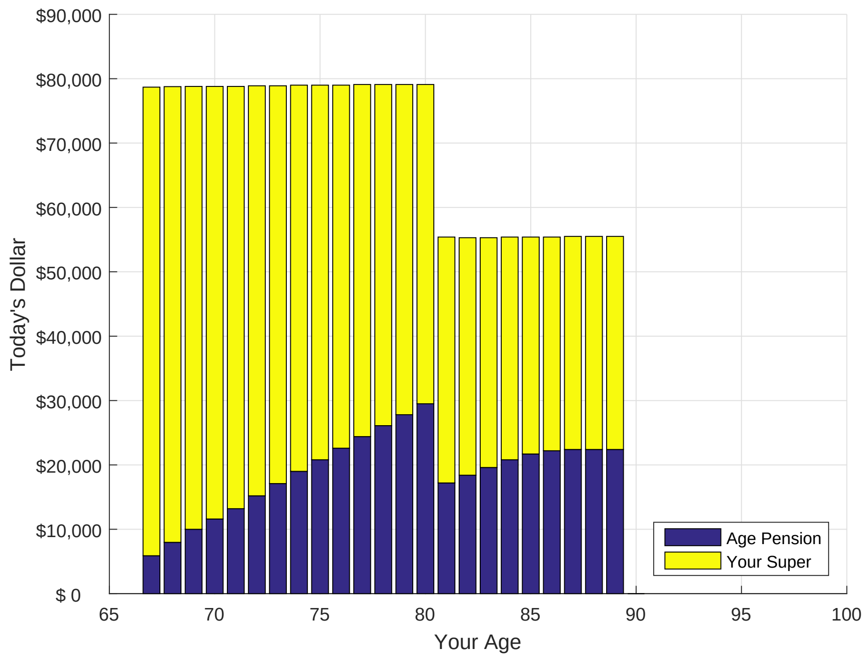

Figure 10: Combined household of a male and a female both aged 65 with a total superannuation balance of $1m and assume household spending cuts by 30% when only one is alive. Note that in a combined household scenario the proportion of each member's superannuation balance does not make a difference so this figure is the same as Figure 5. For the convenience of comparison, we show it here again.

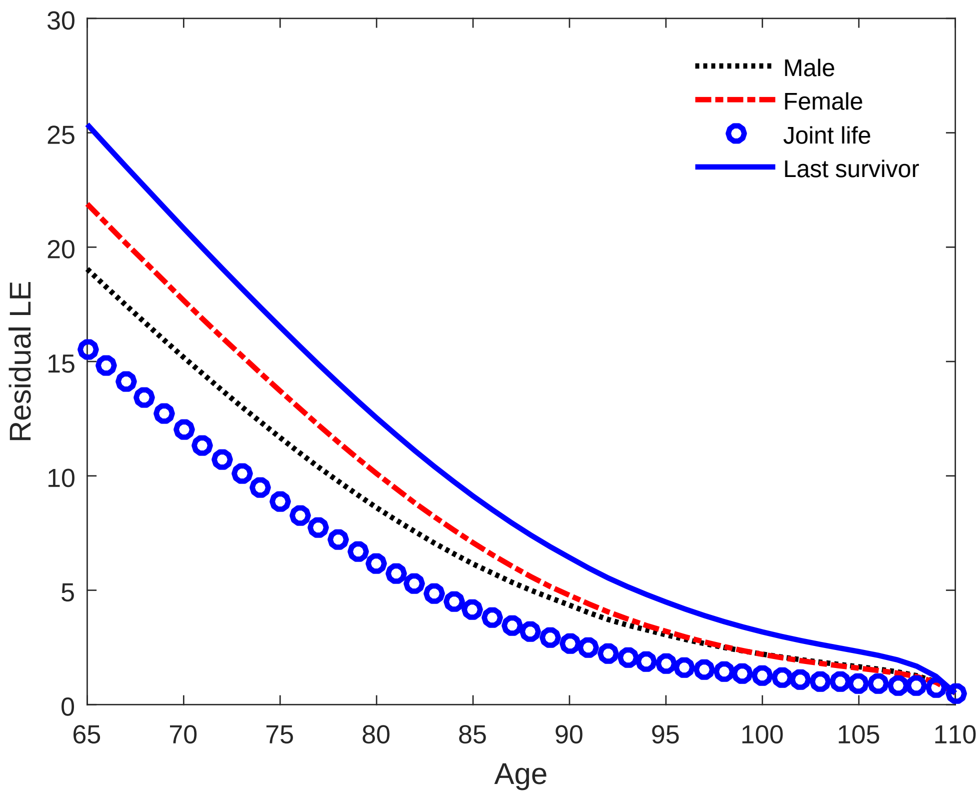

Figure 11: Residual life expectancies for joint lives of the same age.

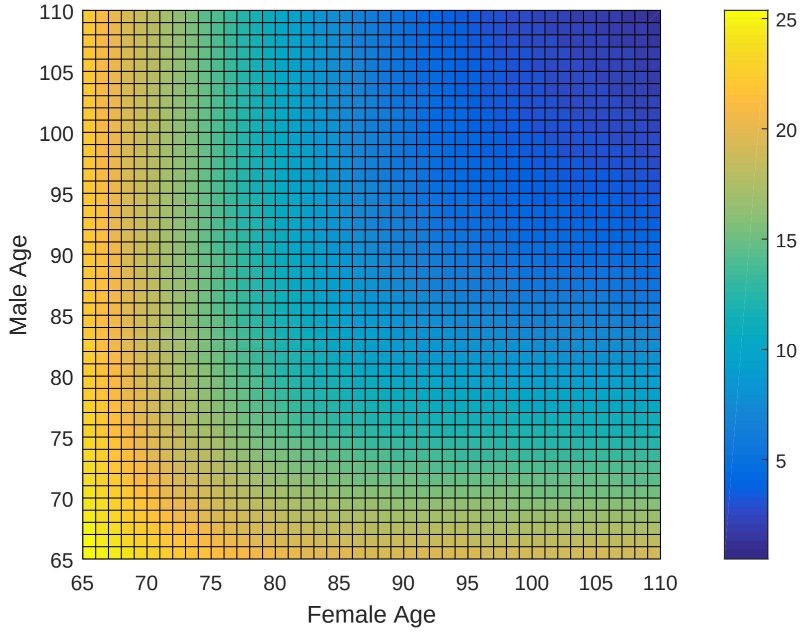

Figure 12: Residual last survivor life expectancies for joint lives of different age combinations.

Appendix 2

Trustee's Utility Function for Default Fund Design

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Abstract

The "retirement challenge" represents the need for industry and policymakers to focus on the efficient delivery of retirement income streams. This compares with the historic focus on accumulation balances. Amongst many challenges we need to address longevity risk, acknowledge that a more certain outcome is valuable and that any residual account value at death has (at least) some value. The Government recently announced its support for the development of comprehensive income products for retirement (CIPR) and the industry will likely be shaped towards this direction. This creates significant challenge for industry and policymakers. For instance how do we compare the range of possible outcomes from two competing fund designs? One way to do this is to shift our focus from pure financial outcomes to the level of satisfaction associated with the range of possible outcomes measured by a representative utility function. To assist industry and policymakers, a working group has spent over a year developing the 'Trustee's Utility Function for Default Fund version 1' (TUFv1). This paper details the design considerations investigated by the working group and the final TUFv1. We illustrate with examples how the utility function can be used to address fund design and policy issues.

Keywords: utility function, default super fund design, retirement outcomes, retirement income, longevity risk, investment risk, CIPR

PRELIMINARY WORKING PAPER

PART I: Background

1 Introduction

1.1 Objectives of superannuation system

The Government recently agreed on the recommendations made by the Financial System Inquiry (FSI) to enshrine the objective of the superannuation system in legislation. This is aiming to provide guidance to the industry for a better superannuation system with considerations around fairness, adequacy and dignity. Different parties such as the Treasury, the Association of Superannuation Funds of Australia (ASFA), and the Australian Institute of Superannuation Trustees (AIST) stated their objectives for the Australian superannuation system in slightly different ways. However, it is clear that the objectives are all around sustainable retirement income.

- **The treasury adopted FSI's view:** provide income in retirement to substitute or supplement the Age Pension.
- **ASFA's view:** help achieve the best retirement outcomes for members of super funds through the development of good public policy and industry best practice
- **AIST's view:** provide equitable, adequate and sustainable retirement incomes by considering the relationship with the Age Pension and the health care system.
- **Super system review:** enhance retirement incomes with a whole of life focus and include a retirement income stream product chosen by the trustee to address longevity, inflation and investment risks.

1.2 Evolution of superannuation system

The superannuation industry is changing its focus in a number of different areas highlighted below:

- Shift of focus from lump sum benefit to retirement income stream
- Shift of focus from managing investment risk and return to managing distribution of retirement outcomes
- In addition to investment risk, other risk factors such as mortality risk, longevity risk, liquidity risk, sequencing risk and inflation risk are also key risks to manage for retirement.
- As such the universe of retirement solutions broadens to include:
  - Investment solutions: investing in different asset classes and or pre-mixed diversified options offered by trustees or investing through Self Managed Super Fund (SMSF) whichever meet members' investment needs.
  - Mortality/Longevity solutions: purchasing products that provide longevity protection to reduce the risk of members outliving their savings before death. Possible products can be life annuities and longevity pooling products which are currently offered in Australia.
  - Default retirement solutions: could be a mix of investment and mortality/longeivty solutions mentioned above recommended based on broad assumptions for members. The Government announced its support to the development of comprehensive in-

come products for retirement (CIPR) and will facilitate trustees pre-selecting these products for members.

– Advised retirement solutions: getting tailored advice from financial advisers for appropriate income drawdowns in retirement based on personal situation.

– Personal retirement solutions: it is between the default and advised retirement solutions where known personal information is used for designing better retirement solutions.

To manage retirement outcome for members, we need guidance as to how we should trade off the level of retirement income against its certainty (how certain can they receive the nominated income stream each year suggested by the trustee) and its variability (the range of possible outcomes due to variability in investment return and mortality outcomes etc.). To do this, we need to shift our focus from pure financial outcomes to the level of satisfaction associated with the range of possible outcomes, we can measure this level of satisfaction by using utility functions.

1.3 Designing retirement income strategies

Retirement income can be measured in absolute dollar amounts, by replacement rates[1] or as a comparison to certain living standards. There are several ways of designing optimal retirement income strategies based on retirement outcome projection techniques:

- Seek to maximise the expected retirement outcome level measured in either absolute dollar amount or by replacement rates. In this case, we do not consider setting a target for income and the year by year fluctuation of the retirement outcome is not managed. This leaves us open to the risk of not managing for the risk of unacceptable outcomes.

- Determine an appropriate target and seek to minimise the expected failure rate with respect to achieving that target. For example, measuring the number of years that a member will experience income shortfall below a chosen target. There are different ways of choosing the appropriate target. It can be achieving 70% replacement rate for retirement or achieving ASFA modest or comfortable living standards. An issue with this approach is that we ignore the consideration of how far we may fall below the target in some scenarios. In this case, we do consider setting a certain target for income but the year by year fluctuation of the retirement outcome is still not managed.

- Seek to maximise the expected utility of retirement outcomes. A utility function is used to reflect the preferences of individuals, and it effectively puts values ('scores') on the retirement experience of a set of cash flows in retirement. A utility function places higher values on favourable outcomes while marking down poor outcomes such as a consistently lower level of retirement income or one-off hit. Unlike the previous two methods, the utility function approach is capable of balancing different aspects of retirement outcomes and allowing for subjective adjustments to how the members value different outcomes. For example, bequests may be considered as an important aspect in retirement solutions. A utility function can be designed to accommodate these outcomes.

[1]**Replacement rate:** the proportion of a member's current employment income that the current value of the member's superannuation benefits would represent if paid to the member as an income stream over the member's retirement.A commonly expressed aspiration for adequacy is for a superannuation balance large enough to provide an income stream (including capital drawdown) of around $60 - 70\%$ of pre-retirement income over a 25-30 year period.

2 Background on utility theory

2.1 What is a utility function?

Utility functions are mathematical functions that are designed to reflect the preferences of an individual. They can reduce the dimension of problems to something that is easier to measure and compare through ranking alternatives according to their utility outcomes. They can be very useful in objectively addressing important questions such as comparing investment opportunities, constructing portfolios, designing and comparing products and assessing the value of advice. A higher utility outcome is always preferred over a lower utility outcome. In theory, investors make choices consistent with maximising the expected value of the utility function if we consider a range of possible utility outcomes accounting for risk factors such as investment returns and mortality experience. Utility can be defined over either wealth or consumption over time, and can also be defined over a bundle of items. The use of a utility function involves a number of consumers' behavioural assumptions such as their risk preference, their reaction to change in economic factors and their wealth level.

In a perfect world we would construct individual retirement plans for each member with a perfect understanding of their preferences. There are two significant hurdles we face in practice. The first one is the administrative restrictions which mean that members are in a pooled environment; we therefore need to make board assumptions on the 'average' member which might not truly reflect individual preferences. The second one is that academic literature suggests that even if the administrative restrictions did not exist, it is still very hard to accurately elicit preferences of individuals. One example could be the poor processes for risk profiling in the financial planning industry. General questionnaires on risk preferences are put forward to clients but these results do not convert well into an economic measure of risk aversion.

For a default fund we would assume a sensible paternal utility function appropriate for a trustee to assume on behalf of its membership. This could be considered by all super funds in the design of their default funds. However they could 'opt-out' and develop their own version if they believe they have greater insight into the preferences of the members in their fund. We suspect two types of 'opt-out' from the other super funds:

- Homogeneous opt-out: the super funds believe their members possess very different characteristics in risk aversion, bequest etc. compared to our default utility function.
- Heterogeneous opt-out: the super funds believes the heterogeneity of their members' characteristics can not be represented by one single utility function from our default utility function. This would be the case if all members received financial advice and preference assessments.

Once established, a utility function becomes our objective beacon for guiding our forward looking decisions in the retirement outcome space. It can then be both an ex-ante guide for us to recommend retirement strategies and an ex-post basis for performance assessment of these recommendations over time. There are a number of aspects to be considered in designing a utility function:

- The utility function is designed from the trustee's perspective and as a result it reflects existing regulations and the spirit of proposed regulations.
- We assume no unique insights into our members beyond their age, gender, balance and contribution levels.

- A trustee is obliged to make paternal decisions that they believe is in the best interests of members. One output of this is default fund design. A member always has the right to opt out of the default.
- The utility function needs to achieve its purpose with minimal complexity and it should be broadly understandable by an appropriately well resourced super fund likely with guidance from a consultant.
- Highly debatable assumptions used in constructing the utility function need to be avoided.

Given all of this we name the function the "Trustee's Utility Function for Default Fund Design v1". We will call this "The Trustee's Utility Function version 1 (TUFv1)" for short. The v1 acknowledges that we believe this function should be reviewed periodically as the industry continues to evolve, particularly from an IT and administration perspective.

2.2 Is the expected utility theory appropriate for the design of default funds?

Expected utility is a criterion as a rule of choice in the presence of risky outcomes. The two big risks to manage as a super fund for members are the investment returns and mortality outcomes. Given the range of possible investment outcomes and mortality outcomes, the resulted range of utility outcomes can be calculated. The expected utility is then the sum of each utility outcome weighted by the likelihood of these outcomes occurring. If a simulation technique is used to generate a large enough range of risk scenarios, then the expected utility can be estimated as the average of all the utility outcomes. Figure 2.1 illustrates the calculation of expected utility.

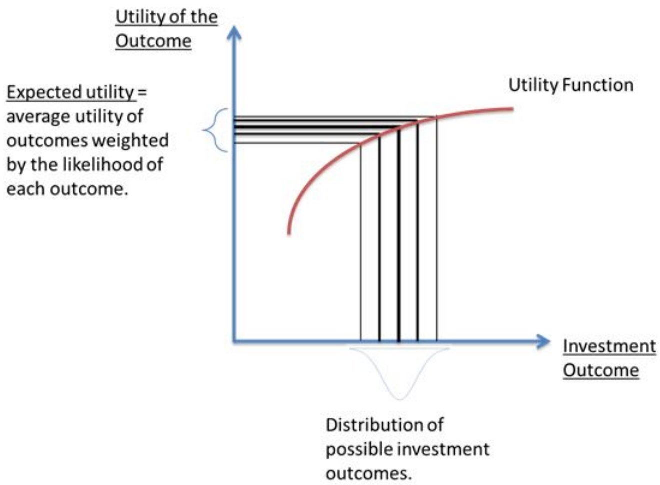


Figure 2.1: Illustration of calculation of expected utility

The expected utility theory is based on four axioms which define a rational decision maker. They are:

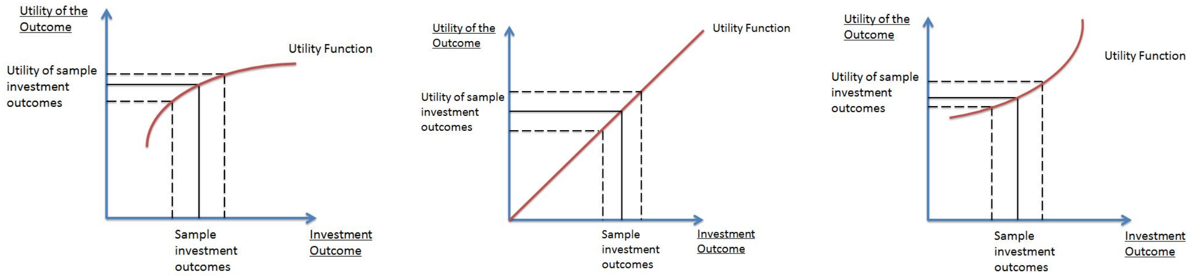
- Completeness: an individual can always decide between two alternatives.
- Transitivity: an individual can decide consistently
- Independence: two gambles mixed with a third one maintain the same preference order as when the two are presented independently of the third one.

- Continuity: if an individual has clear preferences over three gambles, then there should be a possible combination of the most and the least preferred options in which the individual is then indifferent between the mix and the one which sits in the middle.

Expected utility theory is only appropriate when these four axioms hold and the individual is considered to be a rational decision maker under this definition. However, there have been some studies such as Starmer (2000) that revealed empirical evidence of the violation of these axioms by individual decision maker. The question is then whether it is still appropriate for us to use the expected utility theory to design the default funds for our members if not all of them are rational decision makers? We believe the answer is Yes and this is because if we assume the role of a trustee in providing the default retirement products is to act on behalf of their members' best interest in the design process. We believe this assumption is sufficiently valid as the trustee will be making the 'rational' decision on behalf of the individual members who may not always make rational decisions.

2.3 Types of utility function based on risk preferences of investors

Assume $u(w)$ is the mathematical formulae representing the utility function of an individual with wealth (outcome) equal to w. An individual can be identified as risk averse, risk neutral and risk loving depending on the shape of their utility curve.



(a) Risk averse (b) Risk neutral (c) Risk loving

Figure 2.2: Types of risk preference of investors

Table 2.1: Comparing types of risk preference of investors

Risk averse	Risk neutral	Risk loving
An investor prefers a low level of risk on his investments. They will accept a higher level of risk only if the increase in expected returns is sufficient.	An investor is indifferent between choices with equal expected payoffs regardless of the riskiness of each choice	An investor seeks higher risk investments. They will only accept lower risk if it is accompanied by a sufficient improvement in expected returns.
Concave utility curve	Linear utility curve	Convex utility curve
Marginal utility is decreasing as wealth is growing.	Marginal utility is unchanged as wealth is growing.	Marginal utility is increasing as wealth is growing.
Marginal utility is lower for higher outcome.	Marginal utility is constant for all outcome levels.	Marginal utility is higher for higher outcome.

Lower expected returns are usually associated with lower risk investments and higher expected returns are associated with higher risk investments. As most investors expect to be compensated for taking on additional risk, they are considered to be risk averse investors. As a result, we want to focus on exploring risk aversion in more detail.

The classic mean-variance optimisation framework is useful for understanding the behaviour of investors with different risk preferences. In the mean-variance construct investors seek to maximise expected return for a selected level of risk or minimise risk for a targeted expected return. A straight line starting at the risk-free rate tangent to the Efficient Frontier is known as the Capital Market Line (CML). The point on the Efficient Frontier tangent to the CML is known as the Market Portfolio (M). The theory (Markowitz, 1952 and Tobin, 1958) is that all investors should invest along the CML, a portfolio combining the market portfolio (M) and cash (either invested or borrowed).

In a utility maximising framework we can make the following observations, relative to the Market Portfolio (M):

- All investors would actively seek a portfolio which would move them north of M.
- A risk neutral investor is happy to move east of M as they are indifferent to risk. However, in moving east of M they then have an opportunity to move north as well so they would invest on the north-east end of the CML.
- A risk averse investor would be happy to move directly west of M. But a portfolio with such a profile is theoretically unachievable. They may consider a move south-west if the trade-off between lower expected returns and reduced risk was appropriate. A risk averse investor may seek to move east if the additional return available from moving north up to the CML is viewed as sufficient compensation for the extra risk. A risk averse investor will sit on the CML but where they sit will depend on their own risk preference and the slope of the CML (the Sharpe Ratio of M).
- A risk loving investor would be happy to move directly east of M. They may also be prepared to accept a lower expected return in return for higher risk (ie head south-west), but in heading in this direction they would always take the opportunity to subsequently move north and return to the CML. A risk loving investor may consider moving west of M; however they would prefer higher risk, and so to move west they would seek to be compensated with a higher expected return. However moving north-west of M is theoretically not possible. A risk loving investor would sit at the north-east end of the CML.
- Al investors should sit on the CML. A risk averse investor would sit south-west relative to the risk neutral and risk loving investors.

The mean-variance framework implicitly assumes that portfolio returns are normally distributed. However we know that portfolios can be constructed with different shaped return distributions (through the use of option strategies) and that the Age Pension reflects a floor on outcomes. Individuals with different risk preferences have a preference for different outcome distributions. This is demonstrated in figure 2.4. Note that risk neutral investors have no preferences regarding return distributions.

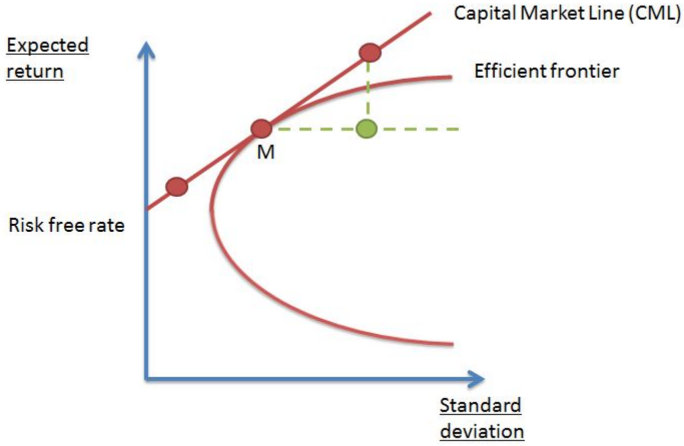
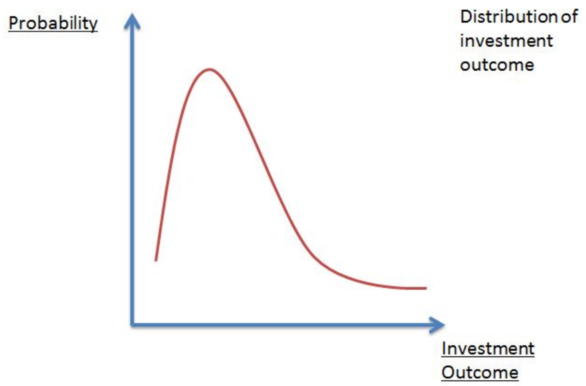
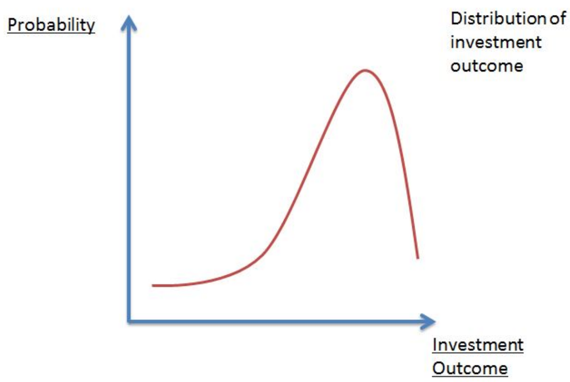


Figure 2.3: Mean-variance portfolio theory and risk preference of investors



(a) Risk averse



(b) Risk loving

Figure 2.4: Preferred distributions for investment outcome based on risk preference of investors

3 Classes of utility function

As risk averse investors are considered to be the majority, we will focus on analysing different classes of utility function in this section for risk averse investors.

3.1 Only focus on terminal wealth (w)

There is a lot of literature which discusses investment strategies in the accumulation phase of defined contribution pension schemes using expected utility maximization of terminal wealth. For example, Boulier et al. (2001), Haberman and Vigna (2002), Deelstra et al. (2003), Devolder et al. (2003), Battocchio and Menoncin (2004),Cairns et al. (2006), Xiao et al. (2007), Gao (2008), and Di Giacinto et al. (2011). This is based on the simplest form of utility function as the input is only one number (the terminal wealth). Risk aversion of an investor can be analysed from two different perspectives:

- Absolute risk aversion (ARA): measures the individual's risk aversion level as the absolute dollar value of their wealth changes. Absolute risk aversion is measured by the negative of the ratio between the second derivatives and the first derivatives of the utility function with respect to wealth as shown below:

$$ARA(w) = -\frac{u''(w)}{u'(w)} \tag{3.1}$$

- Relative risk aversion (RRA): measures the individual's risk aversion level with respect to percentage changes in their wealth. Relative risk aversion is measured by the product of the absolute risk aversion value and the investors' wealth as shown below:

$$RRA(w) = -w \times \frac{u''(w)}{u'(w)} = w \times ARA(w) \tag{3.2}$$

Table 3.1 explains the different types of risk aversions by considering the different attitudes of investors towards investing in risky assets when their wealth changes. Absolute risk aversion focuses on dollar amount changes in risky investment and relative risk aversion focuses on the percentage changes of the portfolio invested in risky assets.

Table 3.1: Types of risk aversions

	Increasing (I)	Constant (C)	Decreasing (D)
Absolute risk aversion (ARA)	IARA: the investor tends to increase the dollar amount invested in risky assets when his/her wealth increases	CARA: the investor tends not to change the dollar amount invested in risky assets when his/her wealth increases	DARA: the investor tends to decrease the dollar amount invested in risky assets when his/her wealth increases
Relative risk aversion (RRA)	IRRA: the investor tends to increase the percentage of his/her wealth invested in risky assets when his/her wealth increases	CRRA: the investor tends not to change the percentage of his/her wealth invested in risky assets when his/her wealth increases	DRRA: the investor tends to decrease the percentage of his/her wealth invested in risky assets when his/her wealth increases

The literature tends to most commonly consider the cases of constant risk aversion (i.e. CARA or CRRA).

3.1.1 Comparison of ARA(w) and RRA(w)

For investors who are CARA, as their wealth increases, the dollar amount they invest in risky assets would not change. This means as a percentage of their wealth, their investment in risky assets actually decreases over time. As a result, CARA investors are also DRRA investors but the inverse is not always true. Similarly, for investors who are CRRA, as their wealth increases, the percentage of their wealth invested in risky assets does not change. This means the dollar amount invested in risky assets actually increases over time. As a result, CRRA investors are also IARA investors but the inverse is also not always true.

3.1.2 Hyperbolic absolute risk aversion (HARA)

Almost all applied theory and empirical work in finance uses some member of hyperbolic absolute risk aversion (HARA) (also known as linear risk tolerance (LRT)) utility functions.

$$u(w) = a\Big(\frac{\rho_w}{1-\rho_w}\Big)\Big(\frac{w}{\rho_w} - \hat{w}\Big) + b \tag{3.3}$$

where ρ_w is the coefficient of relative risk aversion for the HARA utility function defined over wealth (w), a and b are constants as a result of linear transformation. HARA specializes to a number of important utility functions including exponential utility, power utility, log utility, and quadratic utility.

- **CARA or Exponential utility:**

$$u(w) = -e^{-\lambda_w w} \tag{3.4}$$

This is a special case of HARA with $\hat{w} = \Big(\frac{1-\rho_w}{\rho_w}\Big)\frac{1}{\lambda_w}$, $a = \Big(\frac{-\rho_w}{1-\rho_w}\Big)^{-\rho_w}\lambda_w^{1-\rho_w}$, $b = 0$. This gives the following expression of $u(w)$.

$$u(w) = \Big(1 - \frac{\lambda_w w}{1-\rho_w}\Big)^{1-\rho_w} \tag{3.5}$$

$$\lim_{\rho_w \to \infty} u(w) = \lim_{\rho_w \to \infty} \Big(1 - \frac{\lambda_w w}{1-\rho_w}\Big)^{1-\rho_w} = -e^{-\lambda_w w} \tag{3.6}$$

$$ARA(w) = -\frac{u''(w)}{u'(w)} = -\frac{-\lambda_w^2 e^{-\lambda_w w}}{\lambda_w e^{-\lambda_w w}} = \lambda_w \tag{3.7}$$

As $ARA(w) = \lambda_w$ is a constant value and independent of w, we call this type of utility function constant absolute risk aversion (CARA). λ_w is the coefficient of absolute risk aversion. For investors with CARA utility function, when their wealth increases, they will not change the absolute dollar amount invested in risky assets.

- **CRRA:**

$$u(w) = \begin{cases} \dfrac{w^{1-\rho_w}}{1-\rho_w}, & \text{if } \rho_w > 0, \rho_w \neq 1 \text{ (Power utility)} \\ \ln w, & \text{if } \rho_w = 1 \text{ (Log utility)} \end{cases} \tag{3.8}$$

CRRA is a special case of HARA with $\hat{w} = 0$, $a = \rho_w^{-\rho_w}$. In the case of power utility, $b = 0$.

$$RRA(w) = -w \times \frac{u''(w)}{u'(w)} = -w \times \frac{-\rho_w(1-\rho_w)w^{-1-\rho_w}}{(1-\rho_w)w^{-\rho_w}} = \rho_w \tag{3.9}$$

As $RRA(w) = \rho_w$ is calculated to be a constant value and independent of w, we call this type of utility function constant relative risk aversion (CRRA). ρ_w is the coefficient of relative risk aversion. For investors with CRRA utility function, when their wealth increases, they will keep a constant proportion of their wealth invested in risky assets. In the case of log utility, $b = \frac{1}{\rho_w - 1}$. This gives the following expression of $u(w)$

$$u(w) = \frac{w^{1-\rho_w} - 1}{1-\rho_w} \tag{3.10}$$

If we replace $1 - \rho_w = \varphi$ then

$$\lim_{\rho_w \to 1} u(w) = \lim_{\varphi \to 0} \frac{w^\varphi - 1}{\varphi} = \lim_{\varphi \to 0} \frac{\frac{d}{d\varphi}(w^\varphi - 1)}{\frac{d}{d\varphi}\varphi} = \lim_{\varphi \to 0} \frac{w^\varphi \ln w}{1} = \ln w \tag{3.11}$$

$$RRA(w) = -w \times \frac{u''(w)}{u'(w)} = -w \times \frac{\frac{1}{-w^2}}{\frac{1}{w}} = 1 \tag{3.12}$$

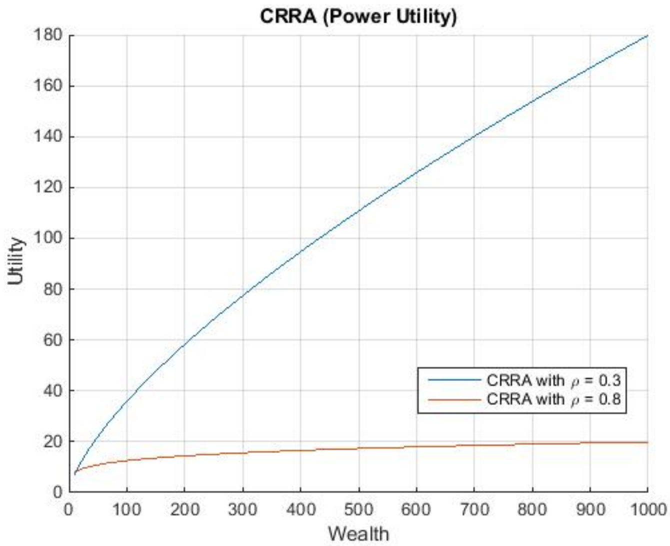


Figure 3.1: Degree of risk aversion and the value of ρ_w

The value of ρ_w determines the degree of risk aversion. This is shown in figure 3.1, where we plot two power utility functions with $\rho_w = 0.3$ and 0.8. A higher ρ_w means a higher degree of risk aversion as the marginal utility from gaining additional wealth is lower.

- **Quadratic utility:** Quadratic utility is a special case of HARA with $\hat{w} < 0$, $\rho_w = -1$, $a = 1$, $b = 0$ and $w < -\hat{w}$. This gives the following expression of $u(w)$.

$$u(w) = -\frac{1}{2}(-w - \hat{w})^2 \tag{3.13}$$

Quadratic utility is neither CARA nor CRRA. To classify quadratic utility function, we derive the equations for ARA and RRA as below.

$$ARA(w) = -\frac{u''(w)}{u'(w)} = \frac{1}{-w - \hat{w}} \tag{3.14}$$

$$RRA(w) = -w \times \frac{u''(w)}{u'(w)} = \frac{w}{-w - \hat{w}} \tag{3.15}$$

The results tell quadratic utility function is IARA and DRRA. Economists avoid using quadratic utility because IARA and DRRA have unrealistic behavioural implications.

3.1.3 Comparing CARA and CRRA

CARA and CRRA are the most common types of utility functions used in academic literature on maximization of expected utility of terminal wealth in DC accumulation phase. Some papers that discuss about using CARA utility function include Devolder et al. (2003) and Battocchio and Menoncin (2004). There are a lot of papers which discuss CRRA as the most widely used utility function. These includes Boulier et al. (2001), Deelstra et al. (2003), Devolder et al. (2003), Cairns et al. (2006), Xiao et al. (2007) and Gao (2008).

It is believed that CRRA describes more logical investors' behaviour than CARA which is consistent with a lot of economics and financial literature. Consider a scenario where an investor has $10,000 and currently invests $5,000 (50%) in risky assets. CARA describes when an investor grows their wealth from $10,000 to $1,000,000, they will still invest only $5,000 in risky assets. However, under CRRA they will always invest 50% which is now $500,000 of their wealth in risky assets.

We prefer CRRA over CARA for the purpose of this paper given the more realistic assumption on investors' behaviour by CRRA. However, there is also a concern about the implications of CRRA particularly as wealth levels become very high. If we continue the above example, it may be realistic to consider that an investor with $1,000,000 would allocate $500,000 to risky assets. But if an investor has $1 billion, would it be realistic to assume that they would invest $500 million in risky assets? Generally there is a concern that the CRRA assumption does not account for a limit to how much absolute financial risk an individual is prepared to take.

We feel that the issues of CRRA in dealing with extremely high wealth levels should not be a limitation for the application under our context. Members in a default fund with extremely high balances are really rare especially with the new $1.6 million transferable pension balance cap proposed in Budget 2016.

3.2 Lifetime consumption (c)

In the history of Australian defined contribution scheme, the industry focus has always been on terminal wealth, namely the lump sum benefit available at retirement. However, the definition of superannuation and the consideration of managing longevity risk guides us to focus on retirement income. As a result, we want to discuss about lifetime consumption problems in addition to terminal wealth.

Lifetime consumption problems take into account multi-period consumption and are based on the expected utility theory. Further exploration is required to determine whether time-separable preference or non-time-separable preference is more appropriate for our context.

Compared to the utility functions which only focus on terminal wealth, ones that account for lifetime consumption are more complicated as they take into account streams of consumption over time.

There are two additional components to consider in a multi-period utility function compared to a single period one. They are the time preference discount factor (also referred to as a patience parameter) and the intertemporal elasticity of substitution:

- **Time preference discount factor:**
  Time preference discount factor considers time value of money. It can be objective if price inflation or wage inflation is used as the discount factor to allow for comparision

of consumption in today's dollar. It can also be subjective if the utility function takes into account personal preferences and an individual's patience to wait and consume in the future.

Wage inflation might be more suitable as a discount factor compared to price inflation due to the fact that wage inflation takes into account increases in living standards in addition to the cost of consumer goods.

- **Elasticity of Intertemporal Substitution (EIS):**
EIS considers more than time value of money, rather it focuses on real return of investments. Basically, it suggests that if market conditions change and investors form different perceptions of the future market return (real), then they will change their preference with regard to deferring consumption into the future. If they expect better future real return, then they might prefer to consume less today to leave a bit more money in the investment to grow for future consumption. The parameter of EIS determines the sensitivity of the investors' consumption behaviour towards changes in their expectations of the market real return. Historical research has shown the value of EIS to be relatively low. In practise, it might be hard to incorporate EIS explicitly with more general utility function specification. We illustrate how EIS interact with the risk aversion parameter in section 3.2.1.

3.2.1 Two-period Problem

The concept of utility function with lifetime consumption can be illustrated using a two period example as shown below.

$$U(c_t, c_{t+1}) = u(c_t) + \beta u(c_{t+1}) \tag{3.16}$$

$$V_t = \mathbb{E}_t[U(c_t, c_{t+1})] = u(c_t) + \beta \mathbb{E}_t[u(c_{t+1})] \tag{3.17}$$

$$\max_{c_t, c_{t+1}} V_t = \max_{c_t, c_{t+1}} \{u(c_t) + \beta \mathbb{E}_t[u(c_{t+1})]\} \tag{3.18}$$

where c_t is the consumption at time t. $u(c_t)$ represents the utility of consumption at time t. $U(c_t, c_{t+1})$ is the total utility of consumption over two periods. $\mathbb{E}_t[\cdot]$ is the mathematical expression for calculating the expected value at time t. In this paper, we use V_t to express the expected utility at time t of future consumptions. The total utility depends on how much the investors consume today at time t and on how much they consume in the future time $t+1$. The parameter β is the time preference discount factor, it captures the weight that investors place on the future relative to today. If $\beta = 1$, then they treat consumptions today and in the future equally. Alternatively, if $\beta < 1$ then they value today's consumption more than the future. This is intuitive and β can be the discount rate of interest representing the time value of money or a subjective discount rate perceived by the investors themselves. Given the role of a trustee, the chosen parameters should reflect objective views to represent the broad membership. $\beta = 1$ is a realistic assumption if we are comparing consumption in real terms. This means inflation rate (price inflation or wage inflation) can be suitable time preference discount factors in this case.

To illustrate the idea of a two-period consumption problem, consider the following scenario. An investor has 2 hundred and is thinking how to spend it over two years so he is most satisfied with the outcome (utility). Assume the investor uses CRRA utility function with $\rho_c = 0.3$ and $\beta = \frac{1}{1+2.5\%}$ discounting for price inflation. ρ_c is the coefficient of relative risk aversion of CRRA utiltiy function defined over consumption. Table 4.2 shows the results of this illustration. The investor's utility is maximised when he splits the 2 hundred to spend 1 hundred each year. Note because $\beta \neq 1$ that the utilities derived under scenario 2 ($c_1 = 80$, $c_2 = 120$) and scenario 4 ($c_1 = 120$, $c_2 = 80$) are not equal.

Table 3.2: Two-period consumption problem illustration

c_1	c_2	$u(c_1)$	$u(c_2)$	$U(c_1, c_2)$	Rank
50	150	20.66	46.23	65.77	4
80	120	29.27	39.34	67.65	3
100	100	34.46	34.46	68.07	1
120	80	39.34	29.27	67.89	2

- **Illustration of EIS:**
For consumption problems involving more than 1 period, the elasticity of intertemporal substitution (EIS) needs to be considered too. EIS measures the change in consumption growth in response to change in expected investment return. We can express EIS in the mathematical formulae below:

$$EIS = \frac{d\ln(\frac{c_{t+1}}{c_t})}{d\ln(1+R)} \tag{3.19}$$

where $c_{t+1} = (w_t - c_t)(1+R)$. If we look at power utility as an example for a 2-period problem, then:

$$\max_{c_t, c_{t+1}} \frac{c_t^{1-\rho_c}}{1-\rho_c} + \beta\mathbb{E}_t\left[\frac{c_{t+1}^{1-\rho_c}}{1-\rho_c}\right] \tag{3.20}$$

Assume R is predetermined, then $\mathbb{E}_t\left[\frac{c_{t+1}^{1-\rho_c}}{1-\rho_c}\right] = \frac{[(w_t-c_t)(1+R)]^{1-\rho_c}}{1-\rho_c}$.

$$\frac{\partial}{\partial c_t}\left\{\frac{c_t^{1-\rho_c}}{1-\rho_c} + \beta\mathbb{E}_t\left[\frac{c_{t+1}^{1-\rho_c}}{1-\rho_c}\right]\right\} \tag{3.21}$$

$$=\frac{\partial}{\partial c_t}\left\{\frac{c_t^{1-\rho_c}}{1-\rho_c} + \beta\frac{[(w_t-c_t)(1+R)]^{1-\rho_c}}{1-\rho_c}\right\} \tag{3.22}$$

$$=c_t^{-\rho_c} - \beta(1+R)c_{t+1}^{-\rho_c} \tag{3.23}$$

$$=0 \tag{3.24}$$

$$\ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\rho_c}\ln\beta + \frac{1}{\rho_c}\ln(1+R) \tag{3.25}$$

$$EIS = \frac{d\ln(\frac{c_{t+1}}{c_t})}{d\ln(1+R)} = \frac{1}{\rho_c} \tag{3.26}$$

The result shows that for power utility, EIS for consumption is equal to the inverse of $RRA = \rho_c$. For other utility functions, EIS can be complex to solve.

3.2.2 Multi-period Problem

It is easy to generalise the expression for lifetime utility with multi-period consumption if we assume the investors have a time additive utility function of the form below. Non-time-separable utility function will be briefly discussed in section 3.2.4.

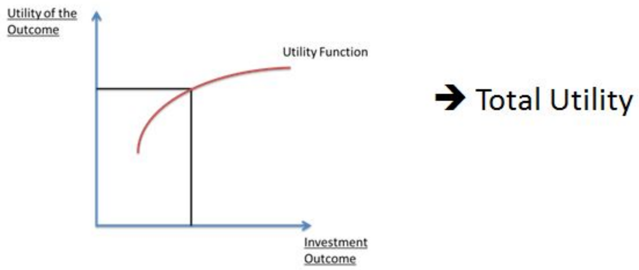
$$U(c_t, c_{t+1}, \ldots, c_T) = \sum_{\tau=t}^{T} \beta^{\tau-t} u(c_\tau) \tag{3.27}$$

$$V_t = \mathbb{E}_t[U(c_t, c_{t+1}, \ldots, c_T)] = \mathbb{E}_t\left[\sum_{\tau=t}^{T} \beta^{\tau-t} u(c_\tau)\right] \tag{3.28}$$

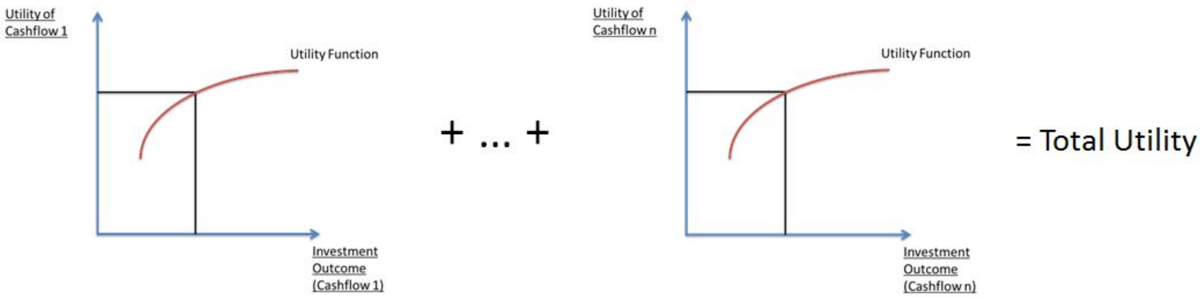
$$\max_{c_t, c_{t+1}, \ldots, c_T} V_t = \max_{c_t, c_{t+1}, \ldots, c_T} = \mathbb{E}_t\left[\sum_{\tau=t}^{T} \beta^{\tau-t} u(c_\tau)\right] \tag{3.29}$$

Figure 3.2 illustrates how multi-period utility outcome is calculated by the summation of a series of single-period utility outcomes.

Figure 3.2: Utility of multi-period cash flows



(a) Utility of a single cash flow



(b) Utility of a collection of cash flows

3.2.3 Specification in addition to wealth and consumption

It is considered that in addition to consumption, people might place value on other aspects such as bequest motive and leisure. In this case, there will be additional terms to be included in the utility function.

Bequest motive:
We are interested in considering bequest motive for two reasons:

- Reversionary to spouse
- It is reasonable to assume that people may place a value on bequest

The following shows an example of the time-separable power utility at time zero discussed in Cocco, et al. (2005) that considers lifetime consumptions and bequest motive. We redefine the utility function at time t in the form below:

$$U(c_t, c_{t+1}, \dots, c_T) = \sum_{i=0}^{T-t} \beta^i \Big\{ {}_ip_t \ u(c_{t+i}) + {}_{i-1|}q_t \ v(b_{t+i}) \Big\}, \tag{3.30}$$

where

$$u(c_t) = \frac{c_t^{1-\rho_c}}{1-\rho_c}$$

and

$$v(b_t) = k\frac{b_t^{1-\rho_c}}{1-\rho_c}.$$

$\rho_c > 0$ measures the level of risk aversion in the sense that as ρ_c increases the person indeed exhibits a more risk averse behaviour. b_t is the level of bequest at time t if the person dies between $t-1$ and t. k is a measure of the strength of bequest motive. $k > 0$ indicates the person will always keep some liquid assets to bequest. ${}_ip_t$ is the probability of being alive at time $\tau + k$ conditional on being alive at time t. ${}_{i-1|}q_t$ is the probability of dying between time $t+i-1$ and $t+i$ conditional on being alive at time t.

At time t, the objective is to maximise the expected present value of utilities through the lifetime. This is expressed in the following equation:

$$V_t = \max_{c_t, c_{t+1}, \dots, c_T} \mathbb{E}_t \left[\sum_{i=0}^{T-t} \beta^i \Big\{ {}_ip_t \ u(c_{t+i}) + {}_{i-1|}q_t \ v(b_{t+i}) \Big\} \right] \tag{3.31}$$

where $\mathbb{E}_t$ is the expectation operator.

Lockwood (2014) separates the consideration of risk aversion over consumption and bequest by including a threshold consumption level in the bequest motive function. The bequest motive function is:

$$v(b_t) = \left(\frac{\phi}{1-\phi}\right)^{\rho_c} \frac{\left(\frac{\phi}{1-\phi}c_b + b_t\right)^{1-\rho_c}}{1-\rho_c}, \tag{3.32}$$

where $c_b \geq 0$ is the threshold consumption level that measures the degree to which bequests are considered as luxury goods (Ameriks et al., 2011; Ding et al., 2014). Without considering any uncertainty, c_b is the level of consumption that makes the member indifferent to 1 dollar additional consumption or 1 dollar bequeath, i.e. if the member's consumption is below this level, they do not want to leave any bequests. Smaller values of c_b means members would want to leave bequest at a lower rate of consumption. $c_b = 0$ means preferences over consumption and bequests are homothetic and members are equally risk-averse over consumption and bequests. This gives the same expression as the bequest function in Cocco et al. (2005).

$\phi \in [0, 1)$ captures the strength of the member's bequest motive when bequest has kicked in (i.e. the wealth level is high enough to allow the member to consume at least c_b). The larger the value of ϕ, the stronger the member's bequest motive.

Family state and leisure:
Hubener, et al. (2013) consider the joint utility of couple, and the trade-off between leisure and consumption in their model. This gives interesting insight into how people place value on lifestyle in addition to spending. Consider the mix of employment types as people choose to work part time or full time. They forgo some income streams for more time spent with family. From a trustee's perspective, the focus would be more on the trade-off between consumption and bequest as it is very hard to quantify the value of leisure for our members and take that into account when we design the members' default fund. Please refer to appendix 5.1 for the technical details of how Hubener, et al. (2013) consider leisure in a multi-period utility function.

3.2.4 Non-time-separable utility function

Habit persistence:
Constantinides (1990) explores an interesting investors' behavioural property called habit persistence by allowing for adjacent complementarity in consumption. This essentially drives a wedge between the relative risk aversion (RRA) of the investors' and their EIS. There are historical evidence such as Hansen and Jagannathan (1991) and Ferson and Constantinides (1991) supporting the concept of habit persistence. The key formulation of a utility function with habit persistence is to measure consumption c_t relative to a time dependent subsistence level of consumption x_t. Constantinides (1990) defines x_t as an exponentially weighted sum of past consumption. This implies that investors place higher utility on gradually changing consumptions over time compared to sudden increase/decrease or more fluctuating consumptions patterns. Practically, it suggests that people can change their consumption habits gradually and find large changes difficult to deal with.

Recursive Utility function:
There is a more complex multi-period utility function with Epstein-Zin preferences called recursive utility function. Unlike the time additive utility function discussed in section 3.2.2, the recursive utility function is non-time-separable. Epstein and Zin (1989) is an extension of the expected utility theory that separates the two consumption preferences: risk aversion ρ_c and the inter-temporal substitutability EIS. This is then more powerful in explaining investors' behaviour as investors' EIS can be unrelated to their level of risk aversion. Please refer to appendix 5.2 for the technical details of Epstein-Zin preferences and the recursive utility function.

3.2.5 Relative or absolute measure

Utility outcomes of consumption can be measured in either absolute or relative terms compared to a consumption floor. Both Kingston and Thorp (2005) and Ganegoda and Bateman (2008) recognise the importance of a consumption floor for retirees.

Some possible options for consumption floors for retirees can be the age pension payments, or lifestyle benchmarks such as the ASFA modest and comfortable living standards.

There are generally two ways to express the relative consumption level:

- Convert the consumption figures into multiples of the consumption floor
- Calculate the difference between the consumption figures and the consumption floor

Kingston and Thorp (2005) and Ganegoda and Bateman (2008) specify a simple HARA utility function in the form below:

$$u(c_t) = \frac{(c_t - c^*)^{1-\rho_c} - 1}{1 - \rho_c} \tag{3.33}$$

where c^* is the consumption floor and $c_t > c^*$. When consumption falls closer to c^*, utility goes to zero and marginal utility approaches infinity. This reflects retirees' motivation to keep their consumption above the predetermined standard of living throughout retirement.

3.3 Aggregation problem

It is straight forward to maximise the utility of the outcome for an individual member. However, when we are dealing with thousands of members and trying to design a default fund that suits the portfolio of our memberships, the aggregation of individual's utility outcomes becomes a problem. At the micro level, conditions like symmetry and separability may hold, while at the macro level they may not. This is known as the aggregation problem. There are a lot of considerations around ethical issues for members' equity and fairness.

An important question to ask is how should we weight individual outcomes in within the total utility of the fund? Should we aggregate each member's expected utility? This means members with higher balance, thus higher income streams, would be weighted higher in the aggregate utility function. Should we equally weight each member's outcome by normalise the expected utility score using the size of their balances?

The answer could be different for different funds depending on their membership profiles. As a result, it might be more appropriate for each super fund to decide which approach to adopt.

Deaton and Muellbauer (1980) discuss about PIGLOG model which was developed to treat aggregate consumer behaviour as if it were the outcome of decisions by a rational representative consumer. We will provide examples and discuss more about the aggregation problem in the next paper.

4 Utility Function Design

In this section, we will discuss a number of considerations for the design of the TUFv1 with our proposed solutions.

4.1 Practical considerations of the TUFv1

1. Is it appropriate to maximise expected utility for optimal design of a default fund?
   Yes.

2. Should we focus on terminal wealth (lump sum) or lifetime consumption (income)?
   Lifetime income approach is more consistent with the super industry's objectives.

3. Should we include bequest motive in addition to consumptions?
   Yes. It is reasonable to assume members place value on bequest especially reversionary benefits to spouse.

4. What members information do we take into account?
   The basic information collected by super fund: age, gender, account balance, contributions.

5. Do we need different functions for subgroup of members?
   Not for the TUFv1 as this will complicate the matter.

6. Consider liquidity constraint?
   Not for the TUFv1 as this will complicate the matter.

7. Consider drawdown control?
   Not for the TUFv1 as this will complicate the matter. The simple analysis below shows

that a big drawdown on members' superannuation investment always has smaller sometime much smaller impact on their retirement incomes.

Table 4.1: Impact of drawdown towards retirement income

Instant drawdown (DD)	Lumpsum = 250,000	Lumpsum = 500,000	Lumpsum = 1,000,000
Age = 67			
5%	2%	3%	4%
10%	5%	6%	7%
20%	9%	12%	15%
30%	14%	19%	23%
Age = 77			
5%	3%	4%	4%
10%	6%	8%	8%
20%	11%	15%	17%
30%	18%	22%	25%
Age = 87			
5%	4%	4%	5%
10%	7%	8%	9%
20%	14%	16%	18%
30%	21%	25%	27%

It is obvious that in retirement, a big drawdown on members' superannuation investment always has smaller impact on their retirement incomes through spreading over multiple periods. The impact of a drawdown on members with lower account balances (in this case the lump sum at retirement) is lower than the ones with higher balances when we taking into account age pension as a buffer. The later the big drawdown occurs at someone's retirement, the stronger the impact on their retirement income will be.

4.2 Parameterisation

1. Time preference discount factor (β)
    - Forecast on price inflation based on CPI data
    - Forecast on wage inflation based on AWOTE data
2. Degree of risk aversion (ρ)

Paper	Utility Function	Parameter	Supporting evidence
Tobin and Dolde (1971)	CRRA defined over life-cycle consumption with borrowing constraint (liquidity constraint)	$\rho_c = 1.5$	Chose 1.5 to fit the observed life cycle savings patterns

Mehra and Prescott (1985)	CRRA defined over life-cycle consumption	$\rho_c < 10$	Concluded based on numerous studies in the past showing ρ_c is always estimated to be smaller than 10. This relies on the estimated 'price of risk' computed by Friend and Blume (1975)
Gourinchas and Parker (2002)	CRRA defined over life-cycle consumption	$\rho_c < 2$	Calibrated by the life-cycle model using Consumer Expenditure Survey data
Chetty (2006)	CRRA defined over life-cycle consumption and labour supply	$\rho_c < 2$	Calibrated by the life-cycle model using data on labour supply behaviour
Schechter (2007)	CRRA defined over life-cycle consumption. Distinct between risk aversion for consumption and wealth. Meyer and Meyer (2005) show that ρ_c for consumption is about 5 times higher than the one for wealth ρ_w	$\rho_c = 1.9$	Calculated using survey data on income and experimental data on bet choice in a risk game for rural Paraguayan households and assuming individuals can not save. Can not save means income and consumption are identical.
		$\rho_c > 1,000$	Same method but assuming individuals can save. This means This gives empirical evidence for narrow bracketing
Yogo (2009)	CRRA defined over life-cycle consumption	$\rho_c = 5$	Calibrated by the life-cycle model based on the Health and Retirement Study
Ameriks et al. (2011)	CRRA defined over life-cycle consumption and end-of-life utility from bequsets	$\rho = 2, 3, 5, 10$	Selected based on other literatures: $\rho_c < 2$ (Gourinchas and Parker, 2002), $\rho > 3$ for a lot of asset pricing studies.
Lockwood (2014)	CRRA defined over life-cycle consumption and bequests	$\rho_c = 2.5$	Calibrated by life-cycle model to match retiree's savings including bequest motives and long-term care insurance choice. Model results range from 2.00 to 2.62.
Janecek (2002)	CRRA defined based on personal investment wealth	$\rho_w = 30$	For average investor based on empirical evidence from a number of experimental results.
		$\rho_w = 20$	For investors with enough experience based on empirical evidence.

$\rho_w > 300$	For investors with little risk-taking experience and distaste for risk endeavors based on empirical evidence.

Table 4.2: Summary of Risk aversion parameter considered in academic literature

- **Age:** Morin and Suarez (1983) argues that older investors are more risk averse than younger investors. Bajtelsmit and Bernasek (2001) and Wang and Hanna (1997) have some debate about this finding. This could be very important to consider since we will be looking at pre-retirement and post-retirement investors.
- **Consumption and bequest:** Lockwood (2014) suggested to separate the consideration of risk aversion over consumption and bequest. It is more realistic to assume bequests are luxury goods and members are less risk-averse over bequests than over consumption. This should be acknowledged in the model.
- **Loss and gain:** Kahneman and Tversky (1979) suggested 'risk seeking' for losses and 'risk aversion' for gains based on their studies on behavioural economics. In our context, a 'loss' means running out of money during retirement and a 'gain' means being able to leave a bequest. They introduced the concept of loss aversion which hypothesizes that individuals place more weight on losses than can be explained by risk aversion. Their experiments found about a 2 for 1 weight for losses versus gains.
- **Constant or changing:** Blanchett et al. (2014) find significant evidence that risk aversion is time-varying. The way risk aversion of the investors change is related to their changing expectations of the future market rather than the experienced market returns. This would have important implications for investors' preference on asset allocations between growth and income assets under different situations.

3. EIS $(\frac{1}{\alpha})$

   - Depends on the risk aversion parameter if time-separable utility function is used.
   - Needs to be estimated separately if non-time-separable utility function is used.

4. Strength of bequest motive (k)

   - It is difficult to determine appropriate values for k with current knowledge as empirical studies are inconclusive on how retirees view bequest (Brown, 2001). One way to determine k is to select some values for k and exam the sensitivity of our results to the changes of k.

5 Appendix

5.1 Utility of family state and leisure

Hubener, et al. (2013) introduce a time budget Θ that separates working hours π_t^i, commuting times $\pi_{t,trav}^i$, child care $\theta_{s,t}^i$ and leisure time l_t^i.

$$\Theta = \pi_t^i + \pi_{t,trav}^i + \theta_{s,t}^i + l_t^i \tag{5.1}$$

where $i = x, y$ denote man and woman in the family; s is the variable for family status that considers never married, married couple, divorced, widowed and the number of children.

The utility funciton at time t then becomes:

$$U(c_t, c_{t+1}, \dots, c_T) = \sum_{\tau=t}^{T} \beta^{\tau-t} \Big(\prod_{j=t}^{\tau-1} p_j \Big) \Big\{ p_\tau \frac{(\frac{c_\tau}{\varphi_s} l_\tau^\theta)^{1-\rho_c}}{1-\rho_c} + k(1-p_\tau) \frac{B_\tau^{1-\rho_c}}{1-\rho_c} \Big\} \tag{5.2}$$

$$V_t = \mathbb{E}_t[U(c_t, c_{t+1}, \dots, c_T)] = \mathbb{E}_t \Big[\sum_{\tau=t}^{T} \beta^{\tau-t} \Big(\prod_{j=t}^{\tau-1} p_j \Big) \Big\{ p_\tau \frac{(\frac{c_\tau}{\varphi_s} l_\tau^\theta)^{1-\rho_c}}{1-\rho_c} + k(1-p_\tau) \frac{B_\tau^{1-\rho_c}}{1-\rho_c} \Big\} \Big] \tag{5.3}$$

$$\max_{c_t, c_{t+1}, \dots, c_T} V_t = \max_{c_t, c_{t+1}, \dots, c_T} = \mathbb{E}_t \Big[\sum_{\tau=t}^{T} \beta^{\tau-t} \Big(\prod_{j=t}^{\tau-1} p_j \Big) \Big\{ p_\tau \frac{(\frac{c_\tau}{\varphi_s} l_\tau^\theta)^{1-\rho_c}}{1-\rho_c} + k(1-p_\tau) \frac{B_\tau^{1-\rho_c}}{1-\rho_c} \Big\} \Big] \tag{5.4}$$

where $l_\tau = \sqrt{l_\tau^x \times l_\tau^y}$ is the effective leisure considering both spouses. φ_s is a scalling factor used for normalized total consumption based on the number of adults and children present in the household. θ measures the preference for leisure with $\theta < 1$ represents declining marginal utility for leisure time.

5.2 Recursive Utility function

Epstein-Zin preference (Epstein and Zin, 1989) is an extension of the expected utility theory that separates the two consumption preferences: risk aversion ρ_c and the inter-temporal substitutability EIS.

We first look at the standard expected utility time-separable preference defined as:

$$V_t = \mathbb{E}_t \Big[\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau) \Big] \tag{5.5}$$

This can be defined recursively as:

$$V_t = u(c_t) + \beta \mathbb{E}_t[V_{t+1}] \tag{5.6}$$

Or equivalently with a scaling of $(1-\beta)$ for consumption at time t:

$$V_t = (1-\beta) u(c_t) + \beta \mathbb{E}_t[V_{t+1}] \tag{5.7}$$

Epstein-Zin preference involves defining recursively $F(\cdot)$ as an 'aggregator' function that maps over current (known) consumption c_t and a certainty equivalent $R_t(V_{t+1})$ of tomorrow's utility V_{t+1}.

$$V_t = F(c_t, R_t(V_{t+1})) \tag{5.8}$$

where

$$R_t(V_{t+1}) = G^{-1}(\mathbb{E}_t[G(V_{t+1})]) \tag{5.9}$$

with $F(\cdot)$ and $G(\cdot)$ increasing and concave. $\mathbb{E}_t[\cdot]$ is the conditional expectation given all the information we know at time t. Most literature considers simple functional forms for $F(\cdot)$ and $G(\cdot)$ as follows:

$$\alpha > 0 : F(c, z) = [(1-\beta)c^{1-\alpha} + \beta z^{1-\alpha}]^{\frac{1}{1-\alpha}} \tag{5.10}$$

$$\rho_c > 0 : G(x) = \frac{x^{1-\rho_c}}{1-\rho_c} \tag{5.11}$$

The inverse of $G(x)$ becomes:

$$\rho_c > 0 : G^{-1}(y) = [(1-\rho_c)y]^{\frac{1}{1-\rho_c}} \tag{5.12}$$

$R_t(V_{t+1})$ can then be expressed in the following form:

$$\rho_c > 0 : R_t(V_{t+1}) = \left[(1-\rho_c)\mathbb{E}_t\left[\frac{V_{t+1}^{1-\rho_c}}{1-\rho_c}\right]\right]^{\frac{1}{1-\rho_c}} = \mathbb{E}_t\left[V_{t+1}^{1-\rho_c}\right]^{\frac{1}{1-\rho_c}} \tag{5.13}$$

$$\rho_c = 0 : R_t(V_{t+1}) = \exp(\mathbb{E}_t[\log(V_{t+1})]) \tag{5.14}$$

$$V_t = F(c_t, R_t(V_{t+1})) = \left[(1-\beta)c_t^{1-\alpha} + \beta\mathbb{E}_t\left[V_{t+1}^{1-\rho_c}\right]^{\frac{1-\alpha}{1-\rho_c}}\right]^{\frac{1}{1-\alpha}} \tag{5.15}$$

$$V_T = F(c_T, 0) = c_T \tag{5.16}$$

In general, ρ_c is the relative risk aversion coefficient and α is the inverse of the elasticity of inter-temporal substitution (EIS) for deterministic variations. A special case is when $\rho_c = \alpha$ or if consumption is deterministic, we will then have the usual standard time-separable expected discount utility with $EIS = \frac{1}{\alpha}$. If $\rho_c > \alpha$, early resolution of uncertainty is preferred when the expected utilities of two lotteries are the same. A higher α gives a lower EIS and thus it means the person becomes less willing to substitute consumption inter-temporally. This means they will be less sensitive in changing their consumption plan when they expect changes in future real interest rates/real investment returns. T is the maximum possible time for the lifetime horizon and V_T is then the utility at time T and $R_t(V_{t+1})$ does not exist.

Bequest motive: If we add in bequest motive, it will lead to the following expression. This is discussed in Horneff et al. (2008), Horneff et al. (2009) and Blake et al. (2014).

$$V_t = F(c_t, R_t(V_{t+1})) = \left[(1-\beta)c_t^{1-\alpha} + \beta\mathbb{E}_t\left[p_t V_{t+1}^{1-\rho_c} + (1-p_t)\left(\frac{k}{1-\rho_c}\right)\left(\frac{B_{t+1}}{k}\right)^{1-\rho_c}\right]^{\frac{1-\alpha}{1-\rho_c}}\right]^{\frac{1}{1-\alpha}} \tag{5.17}$$

$$V_T = F(c_T, 0) = \left[c_T^{1-\alpha} + \beta\mathbb{E}_T\left[\left(\frac{k}{1-\rho_c}\right)\left(\frac{B_{T+1}}{k}\right)^{1-\rho_c}\right]^{\frac{1-\alpha}{1-\rho_c}}\right]^{\frac{1}{1-\alpha}} \tag{5.18}$$

PART II: Trustee's Utility Function v1

1 Introduction

This part shows an illustration of our proposed Trustee's Utility Function v1 (TUFv1) that can be tailored for specific use in the superannuation industry. The developed TUFv1 uses a life-cycle model that takes into account investment and mortality risks.

Some key points of the TUFv1 that need to be kept in mind:

1. Utility function is an appropriate tool to reflect our members' preference.
2. The focus is on lifetime consumption (income) rather than terminal wealth.
3. Bequest is considered to recognise the value placed by members especially on reversionary benefits to spouse.
4. Collectable members information such as age, gender, contribution, and account balance are taken into account in the design of the utility function.
5. Rational behaviour of our members are assumed. Behaviour biases are not part of the consideration.
6. The design of TUFv1 is supported by massive academic literatures.

TUFv1 can be used in the following ways:

1. To compare a number of solutions by ranking them based on their utility scores.
2. To seek optimal solution within a single product. For example, by recommending optimal consumption path.
3. To seek optimal solution across multiple products. For example, by recommending optimal mix of products.
4. To quantify the value of advice by calculating the utility gain of the better retirement strategies.
5. To quantify the utility cost of suboptimal solution.
6. To function as a framework for asset pricing (mostly in academia but not much in industry).

The structure of this paper is as follows. Section 2 describes the model framework and introduces notation. Note that notation in Part II is self-consistent and there can be inconsistency in notation between Part I and Part II. Section 3 briefly outlines the solution technique that is used to maximise lifetime utility. Case studies with numerical results are shown in Section 4. A series of sensitivity analyses are performed and included in the Appendix. The last section proposes recommendations with respect to the life-cycle model to be used in practice.

2 Framework

The utility function is designed in a way to jointly solve for optimal lifetime consumption patterns and allocation to risky assets. This can be represented by O_t which denotes the vector that consists of all choice variables at time t. In particular,

$$O_t = (c_t,\ \omega_t) \quad 0 < t \le T, \tag{2.1}$$

where c_t is the annual consumption level at time t, and ω_t is the weight of wealth invested in risky asset at time t.

The life-cycle utility maximisation problem is:

$$\max_{\{O_t\}_{0\le t\le T}} \mathbb{E}_0\left[\sum_{t=0}^{T} \beta^t \left\{ {}_tp_x \; u(c_t) + {}_{t-1|}q_x \; v(b_t) \right\}\right], \tag{2.2}$$

subject to

$$\begin{aligned} c_t &\ge 0, & (2.3)\\ \omega_t &\in [0,\ 1], & (2.4)\\ b_t &\ge 0, & (2.5) \end{aligned}$$

where x is the inception age of a particular cohort, and $\mathbb{E}_0$ is the expectation operator with respect to time 0 (equivalently, age x). $u(c_t)$ is the utility function defined over consumption at time t:

$$u(c_t) = \frac{c_t^{1-\rho}}{1-\rho}. \tag{2.6}$$

$v(b_t)$ is the utility function defined over end-of-life bequest:

$$v(b_t) = \frac{(b_t)^{1-\rho}}{1-\rho}\left(\frac{\phi}{1-\phi}\right)^{\rho}, \tag{2.7}$$

where b_t is the level of wealth at time t which equals the amount of bequest if the person dies between $t-1$ and t. $\rho > 0$ is the level of risk aversion. Higher ρ means more risk aversion of the investors. ϕ is the strength of bequest motive. Higher ϕ means stronger bequest motive. ${}_tp_x$ is the probability of being alive at age $x+t$ conditional on being alive at age x. ${}_{t-1|}q_x$ is the probability of dying between age $x+t-1$ and $x+t$ conditional on being alive at age x.

The dynamics of wealth is

$$b_{t+1} = (b_t + P_t - c_t)\left(e^{r_f} + \omega_t\left(e^{R_{t+1}} - e^{r_f}\right)\right), \tag{2.8}$$

where P_t is the amount of Age Pension entitlement received at time t if considered, r_f is the annual risk-free interest rate, and R_t is the stochastic return p.a. of risky assets from time $t-1$ to t.

2.1 More Explanation on Utility Function and Bequest Motive

We provide more intuitive explanation on the utility function and bequest motive function we use for the TUFv1.

2.1.1 Utility Function

The utility we adopt is a CRRA utility function of consumption (see Section 3 in Part I for a detailed review of different classes of utility functions). The general pattern of the adopted utility function with respect to consumption is shown in Figure 2.1.

We have the following observations for the utility function:

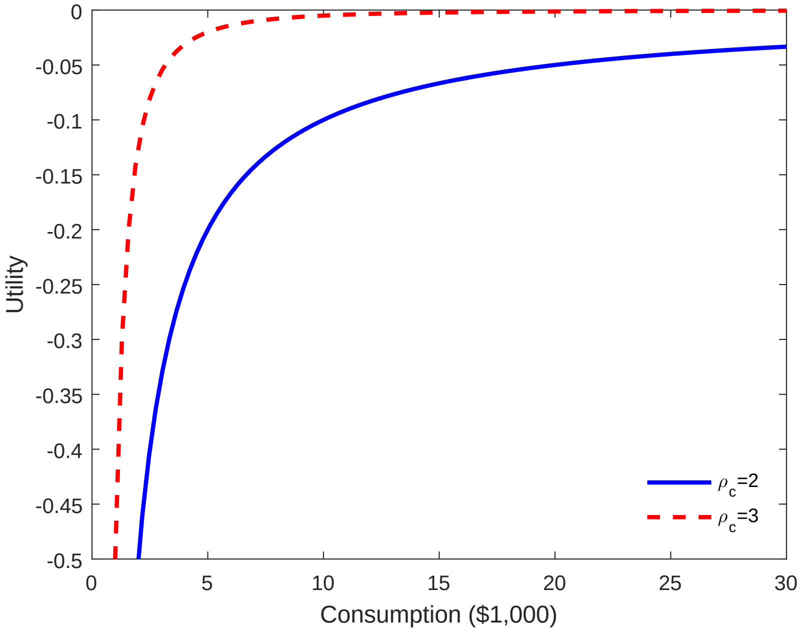


Figure 2.1: Utility function with respect to consumption.

- The higher the consumption level, the higher the utility.
- Utility is a ranking of preferences - the absolute level does not mean anything.
- ρ is called the risk aversion parameter that represents people's conservativeness. For risk-averse people (i.e. $\rho > 0$), the penalty of $1,000 less consumption is higher than the reward of $1,000 dollar more consumption. Table 2.1 shows an example on how utility changes when consumption is $1,000 lower or higher. We see that when the risk aversion parameter is 2, the utility is reduced by 11% when the consumption is lowered from $10,000 to $9,000, where as the percentage increase in utility is only 9% when the consumption is increased from $10,000 to $11,000.

Table 2.1: Percentage changes in utility when consumption changes from $10,000.

	$1,000 Less	**$1,000 More**
$\rho = 2$	-11.10%	9.10%
$\rho = 3$	-24.00%	18.00%

- The higher the risk aversion parameter, people are more afraid of lower consumption. From Table 2.1 we can see that a higher risk aversion parameter causes a more pronounced utility reduction resulting from a lower consumption level compared to utility increase from a higher consumption.
- Time preference is captured by the utility discount factor β. Individuals generally prefer current consumption to future consumptions when other things are equal.

2.1.2 Bequest Motive Function

The bequest motive on terminal wealth is a shifted and scaled function of the adopted utility function. The bequest motive shares the same risk aversion parameter (i.e. ρ) with the utility function. In the following we explain key observations of bequest motive function, i.e. how bequest motive changes with respect to the bequest strength parameter.

- The higher the wealth level (as long as positive), the higher the utility from leaving bequest.
- As the name suggests, a higher bequest strength parameter represents more preference to leaving bequest v.s. consumption. When $\phi = 0.5$, the individual obtains equal utility from annual consumption and from leaving bequest.

2.2 Financial Assets

We have two financial assets included in our framework, including a risk-free asset (cash account) and a risky asset (equity).

We assume that the real risk-free rate is fixed at r_f p.a. for the horizon considered.

Real returns for risky assets are assumed to follows an identical and independent normal distribution, i.e.:

$$R_t \sim N(\mu_R,\ \sigma_R^2).$$

3 Solution Technique for Deriving Optimal Investment and Consumption Solution

Using the Bellman equation, the utility maximisation problem in Equation (2.2) can be expressed in the following recursive equation:

$$V(b_t) = \max_{O_t} \left\{ u(c_t) + \beta \mathbb{E}_t \left[p_{x+t}\ V(b_{t+1}) + q_{x+t}\ v(b_{t+1}) \right] \right\}, \tag{3.1}$$

where $V(b_t)$ denotes the maximised utility based on information up to time t, p_{x+t} is the annual survival probability of an individual aged $x+t$, and q_{x+t} is the annual death probability of an individual aged $x+t$. This recursive equation in essence ensures that the optimum of the optimum is the global optimum.

First-order conditions and envelop conditions are then derived based on the Bellman Equation 3.1. The first-order condition is obtained by equating the first-order derivative of $V(b_t)$ with respect to each of the two choice variables to 0. The envelop condition refers to the optimal $V(b_{t+1})$ in the above equation. Optimal solution is directly coded based on the derived first-order condition. The envelop condition is useful in that it gives us the formula for an important component in the first-order condition.

Starting from the terminal period, we define grid points of wealth dynamics and then perform backward inductions to obtain optimal choice based on first-order conditions and envelop conditions. Once the optimal solution for each grid point of wealth is obtained, forward simulations are then used to demonstrate optimal paths of consumption and asset allocation. The resulting dynamics of wealth and probability of default can then be calculated using these forward simulations and their corresponding optimal choice variables.

4 Case Studies

Our case studies are centred around the account-based pension. We show numerical results for the above proposed life-cycle model with and without taking into account the Age Pension.

Mortality rates are sourced from the Australian Life Tables 2010-12 by Australian Government Actuary[2]. Survival curves and death distributions for males and females are shown in Figure 4.1.

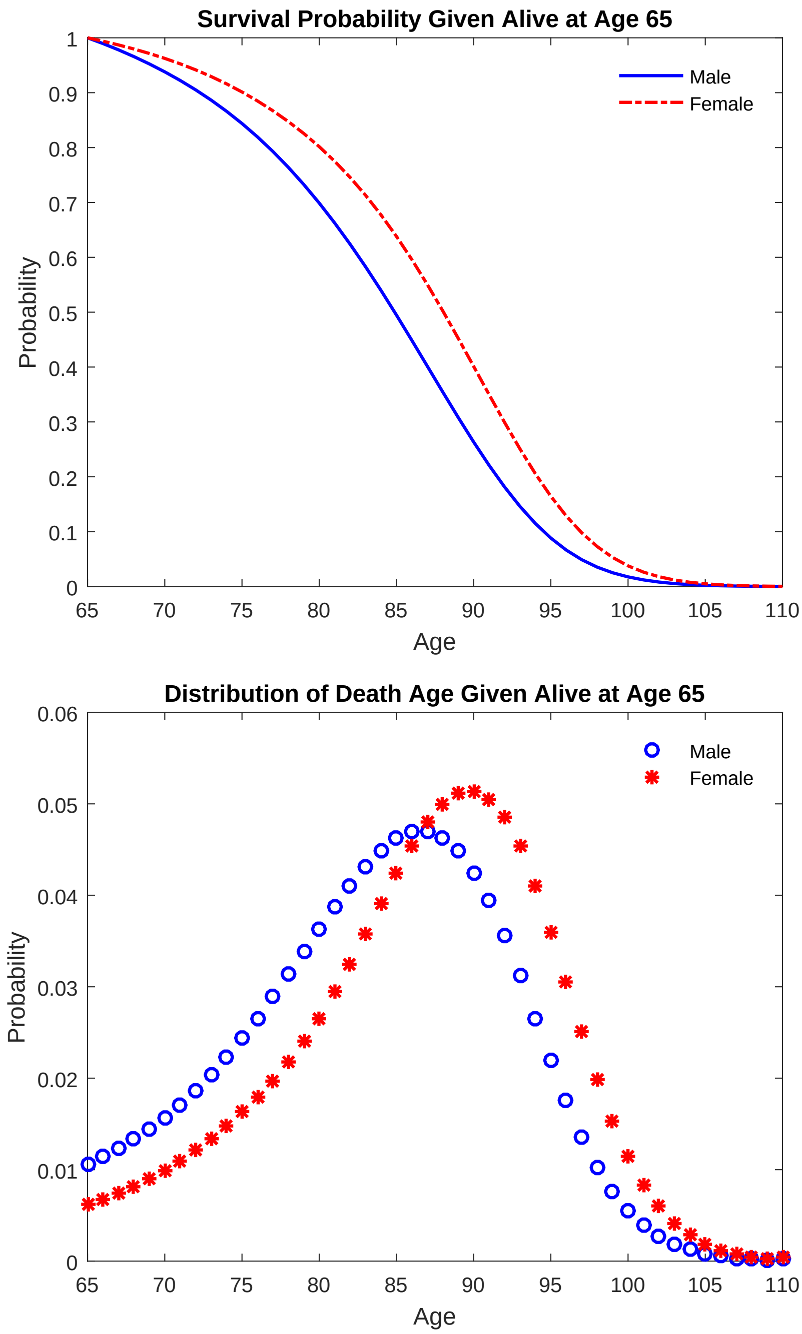


Figure 4.1: Survival curves and death distributions conditional on surviving to age 65.

[2]The Australian Life Tables 2010-12 can be downloaded via `http://www.aga.gov.au/publications/life_table_2010-12/`

For illustration purposes, we only show case study results for males in this section.

4.1 Case 1: Without Age Pension

We start with a scenario where Age Pension is not taken into account. Parameter values are shown in Table 4.1. The optimal average consumption paths, asset allocation, wealth paths,

Table 4.1: Parameter values for base-case analysis. Sources are cited in brackets.

Parameter	Explanation	Value	Source
r_f	Risk-free rate	0.00%	Assumption
μ_R	Mean equity return	5.00%	Assumption
σ_R	Standard Deviation of equity return	15.00%	Assumption
ρ	Risk aversion	5	Ameriks et al. (2011)
β	Utility discount factor	0.96	Ameriks et al. (2011)
ϕ	Bequest motive strength	0.83	Lockwood (2014)
b_0	Initial wealth ($1,000)	500	Assumption
P_t	Age Pension Entitlement ($1,000)	0	Assumption

and propensity to consume (consumption as a proportion of wealth) are shown in Figure 4.2.

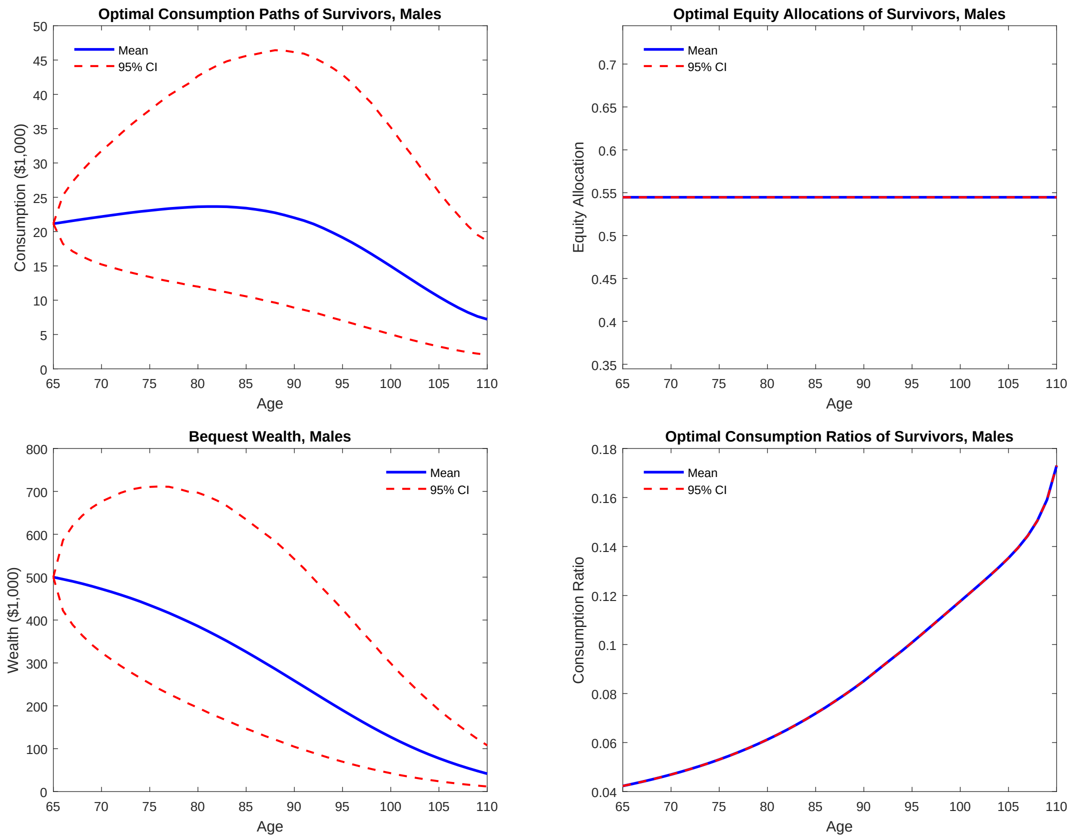


Figure 4.2: Optimal average consumption, asset allocation and wealth paths.

We can see from Figure 4.2 that the optimal allocation to risky asset is a constant proportion, which is 54.47%. This is expected due to the CRRA specification of utility from consumption and from bequest. The optimal average consumption path slightly increases for the first 10 years.

This is because retirees tend to sacrifice their consumption in the first few years for potentially more accumulation in their wealth. The average consumption path starts to decrease from age 85 onward, which is largely due to the consideration of (idiosyncratic) longevity risk.

Another interesting observation is the very stable (as shown by the almost deterministically increasing pattern) propensity to consume, which is calculated as the ratio of consumption to wealth. The consumption ratio starts from 4% (the very traditional rule of thumb Bengen (2004)) at retirement and more than doubles when the retiree reaches age 88.

We also show 6 simulated optimal consumption paths and wealth paths in Figure 4.3. These lines stop sharply due to deaths.

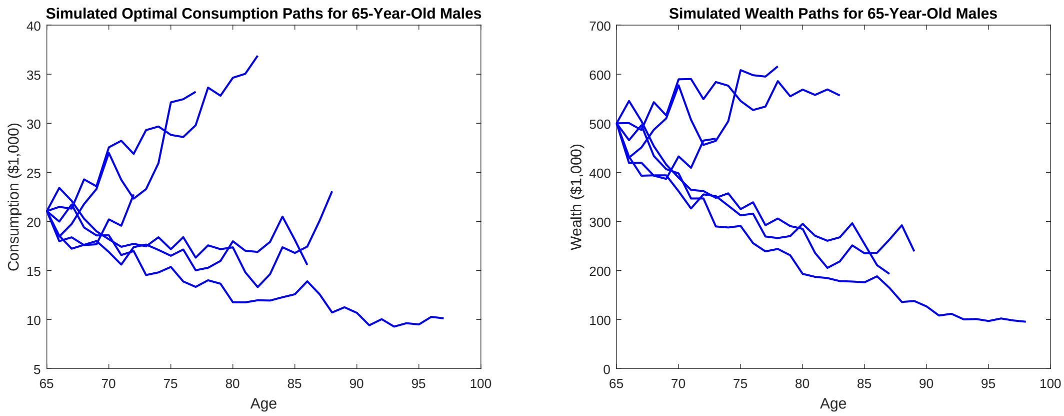


Figure 4.3: Optimal consumption and wealth trajectories for 6 simulated scenarios.

4.2 Case 2: With Age Pension

In this section, we incorporate the Age Pension, based on the Australian Government's eligibility rules. The amount of Age Pension entitlement is means-tested, i.e. depending on the current asset value and future expected incomes. We formulate the following tests to determine the Age Pension entitlement:

- Asset test
$$P_t^A = \max\left(\bar{P} - \tau^A \max\left(b_t - \underline{b},\ 0\right),\ 0\right), \tag{4.1}$$
where P_t^A is the Age Pension entitlement under the asset test, $\underline{b}$ is the asset test threshold for full pension, $\bar{P}$ is the full Age Pension payment rate, and τ^A is the taper rate for asset test.

- Income test under deeming rule
$$P_t^I = \max\left(\bar{P} - \tau^I \max\left(r_1 \min\left(b_t,\ \underline{b}_1\right) + r_2 \max\left(b_t - \underline{b}_1,\ 0\right) - \underline{I},\ 0\right),\ 0\right), \tag{4.2}$$
where P_t^I is the Age Pension entitlement just under the income test, $\underline{b}_1$ is the deeming threshold, below which a lower deeming rate r_1 is assumed and above which a higher deeming rate r_2 is assumed. $\underline{I}$ is the income test cut off point. τ^I is the taper rate for income test.

- Combining the two tests
$$P_t = \min\left(P_t^A,\ P_t^I\right). \tag{4.3}$$

Table 4.2: Age Pension eligibility and payment rates as at 30 June 2016. Single and non-home owner rates are used. Information are sourced from the Australian Government Department of Human Service website.

Parameter	Explanation	Value
$\bar{P}$	Full Age Pension payment rate (p.f.)	\$873.90
τ^A	Taper rate under the asset test	0.0015
$\underline{b}$	Threshold for full pension under asset test	\$450,000
$\underline{b}_1$	Threshold for different deeming rates under income test	\$49,200
r_1	Lower deeming rate	1.75%
r_2	Higher deeming rate	3.25%
$\underline{I}$	Income test cut off point (p.f.)	\$164
τ^I	Taper rate under the income test	0.5

The optimal average consumption paths, asset allocation, wealth paths, and propensity to consume (consumption as a proportion of wealth) are shown in Figure 4.4.

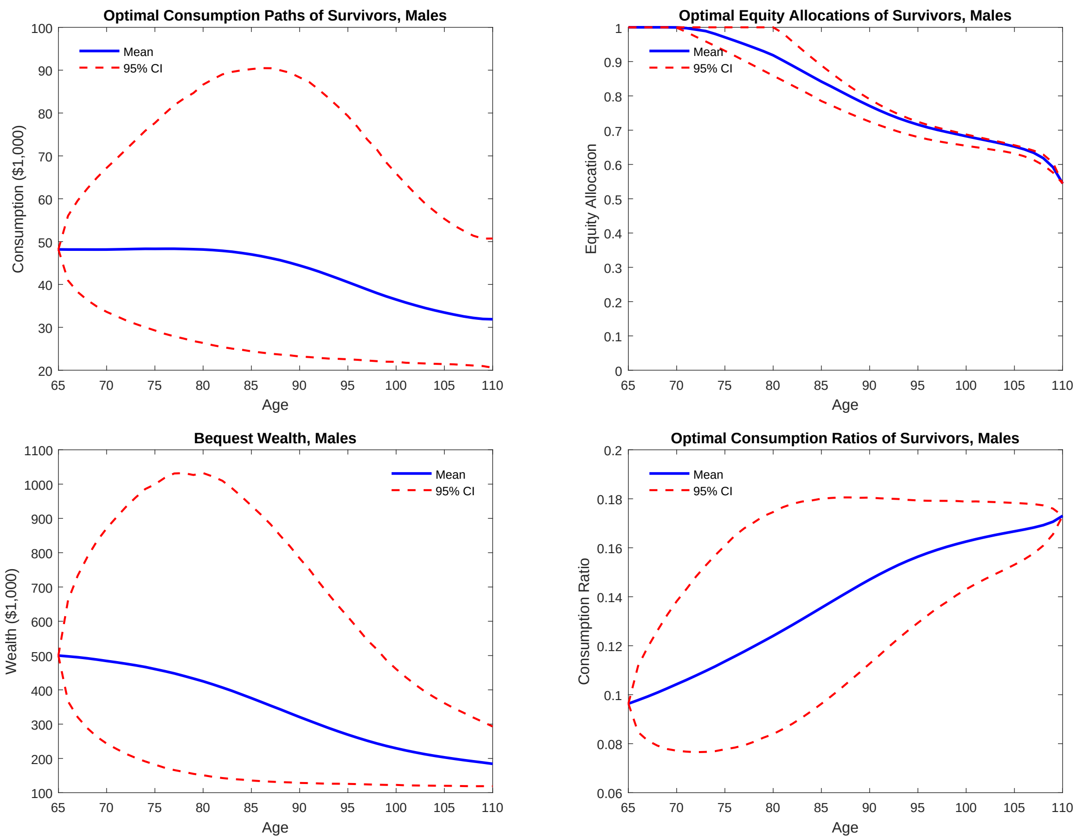


Figure 4.4: Optimal average consumption, asset allocation and wealth paths when the Age Pension is taken into account.

We see a generally higher proportion of wealth that should be invested in equity, especially in early years. The optimal allocation to equity is 100% and starts to decrease in 10 years after retirement. The consumption ratio is significantly higher than the base case where there is Age Pension. The dollar amount of consumption is also higher than in the base case.

A few simulated optimal consumption, asset allocation and wealth paths are shown in Figure 4.5.

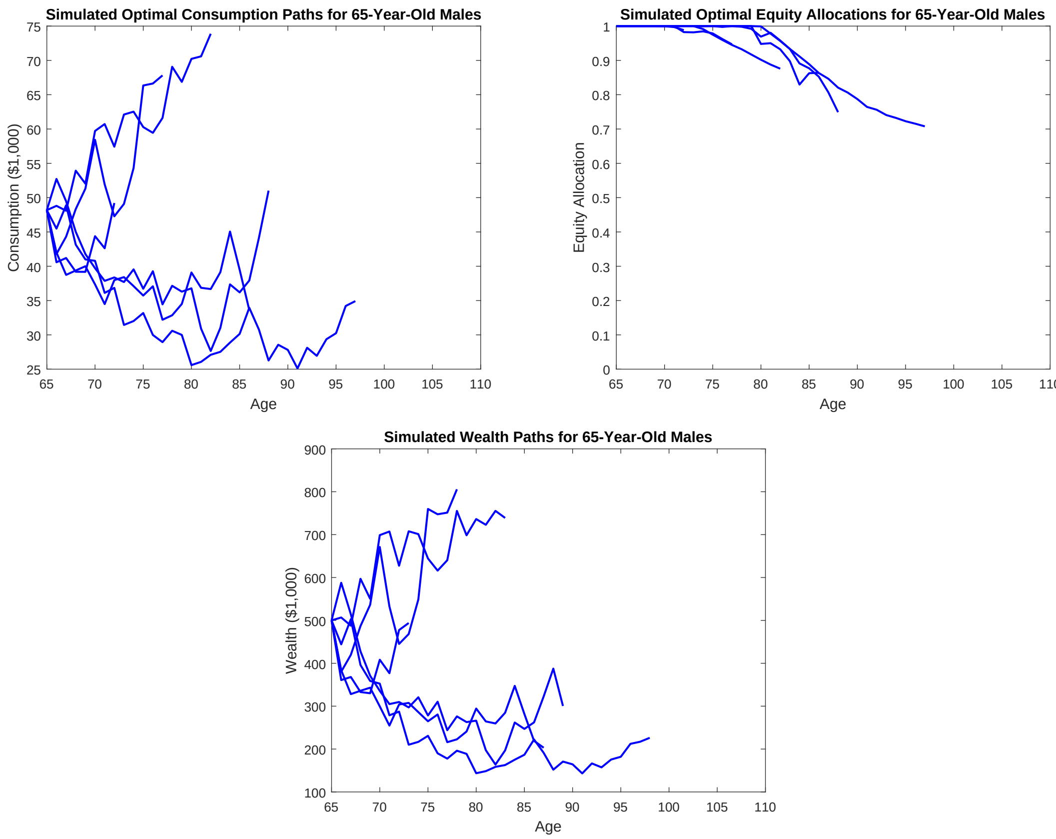


Figure 4.5: Optimal consumption and wealth trajectories for 6 simulated scenarios when the Age Pension is taken into account.

The amount of Age Pension received is shown in Figure 4.6. The left panel shows the average and quantiles of Age Pension entitlements; the right panel displays the amount of Age Pension received for six simulations. The average amount of Age Pension received slightly decreases for the first three years and starts to steadily increase onward. The slight decreasing pattern for the first three years is largely due to the non-linear relationship between Age Pension entitlement and wealth level. As shown in the right panel among the six simulated paths, the magnitude of decreases is much higher than that of increases. After the first three years, the average value of Age Pension entitlement increases because the level of wealth is likely reduced so the probability of receiving Age Pension becomes high.

4.3 Case 3: The Role of Life Annuities

In this case, we investigate the role of life annuities in retirees' retirement planning. We assume that life annuities provide inflation-linked annual benefits and are priced using risk-free interest rate.

We formulate the following tests to determine the Age Pension entitlement:

- Asset test

$$P_t^A = \max\left(\bar{P} - \tau^A \max\left(b_t^p + b_t^a - \underline{b},\ 0\right),\ 0\right), \tag{4.4}$$

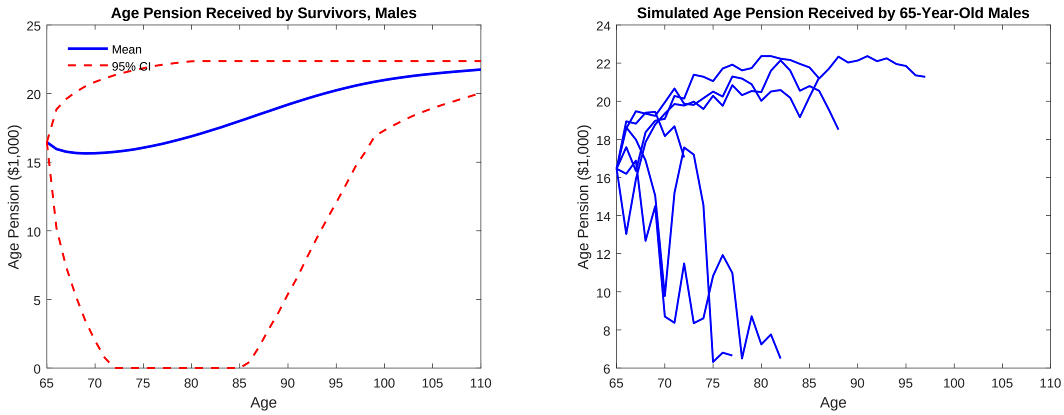


Figure 4.6: Age Pension Entitlements.

where

$$b_t^a = b_0^a - \frac{b_0^a}{\mathring{e}_x} \times t \tag{4.5}$$

b_t^p and b_t^a are the time t value of assets in account based pension and annuity respectively. b_0^a is the purchase price of annuity at time 0. $\mathring{e}_x$ is the life expectancy of the member age x when the annuity is purchased at time 0.

- Income test

$$P_t^I = \max\left(\bar{P} - \tau^I \max\left(r_1 \min\left(b_t^p,\ \underline{b}_1\right) + r_2 \max\left(b_t^p - \underline{b}_1,\ 0\right) + I_t^a - \frac{b_0^a}{\mathring{e}_x} - \underline{I},\ 0\right),\ 0\right), \tag{4.6}$$

where I_t^a is the annuity income at time t and $\frac{b_0^a}{\mathring{e}_x}$ is the deduction amount under the Age Pension rule for annuity.

- Combining the two tests

$$P_t = \min\left(P_t^A,\ P_t^I\right). \tag{4.7}$$

The optimal proportion of wealth that should be annuitised is found to be 0.367. Figure 4.7 shows the level of Certainty Equivalent Consumption (CEC) for different annuitisation ratios.

The optimal average consumption paths, asset allocation, wealth paths, and propensity to consume (consumption as a proportion of wealth) are shown in Figure 4.8.

A few simulated optimal consumption, asset allocation and wealth paths are shown in Figure 4.9.

The amount of Age Pension received is shown in Figure 4.10. The left panel shows the average and quantiles of Age Pension entitlements; the right panel displays the amount of Age Pension received for six simulations. The average amount of Age Pension received shows a similar pattern as in Case 2. A detailed comparison of the Age Pension entitlements between the two cases is shown in Section 4.4.3.

4.4 Comparative Analysis

This section provides a comparative analysis across the three cases.

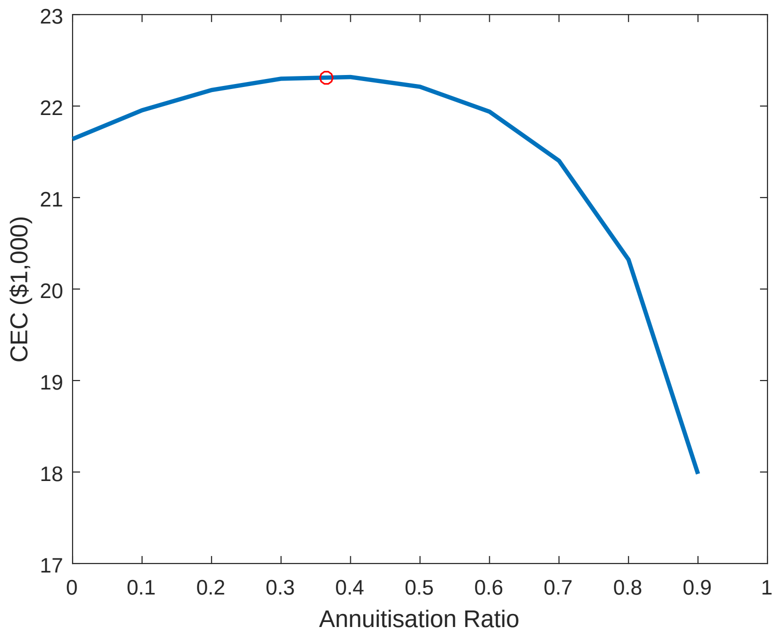


Figure 4.7: Certainty Equivalent Consumption (CEC) for different annuitisation ratios. The highest CEC is marked in red circle.

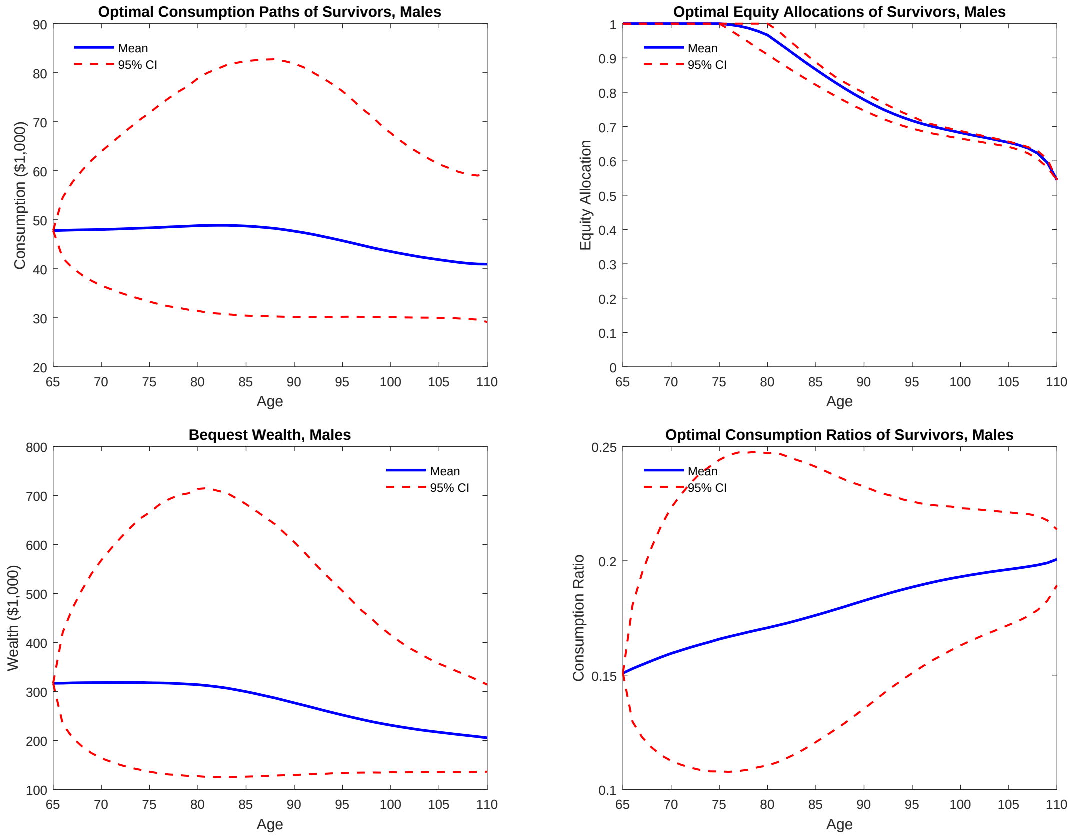


Figure 4.8: Optimal consumption, asset allocation and wealth paths when the Age Pension and life annuities are taken into account.

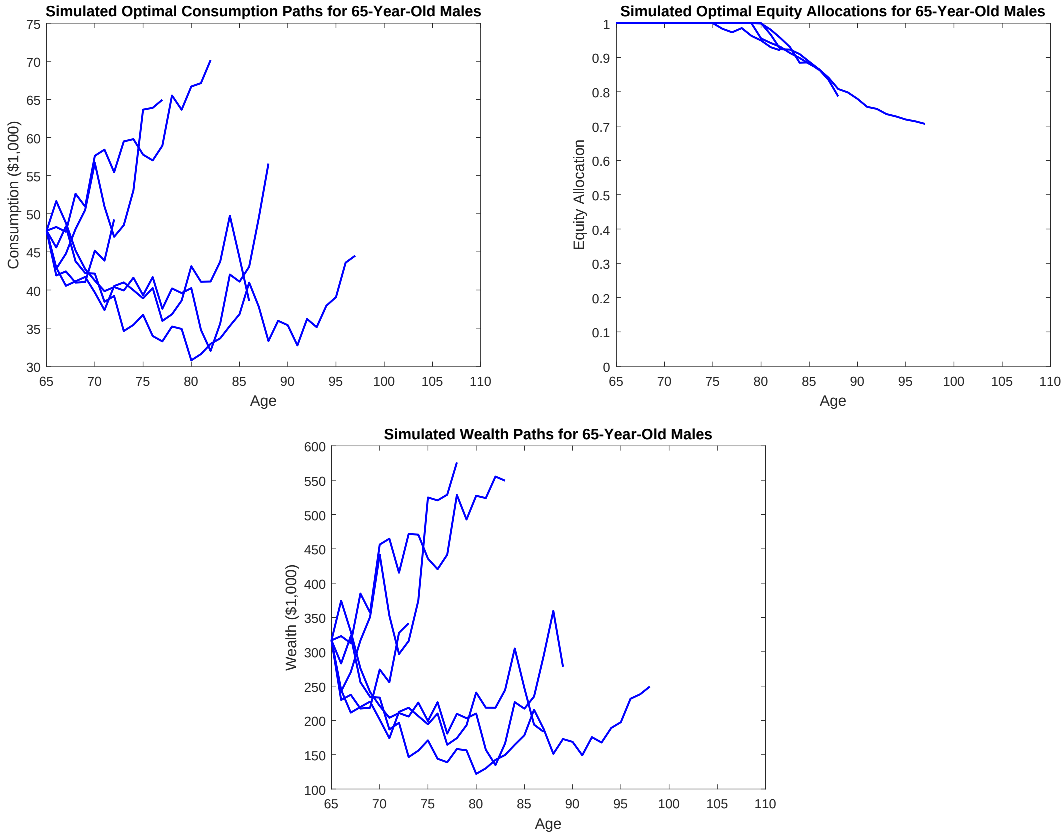


Figure 4.9: Optimal consumption and wealth trajectories for 6 simulated scenarios when the Age Pension and life annuities are taken into account.

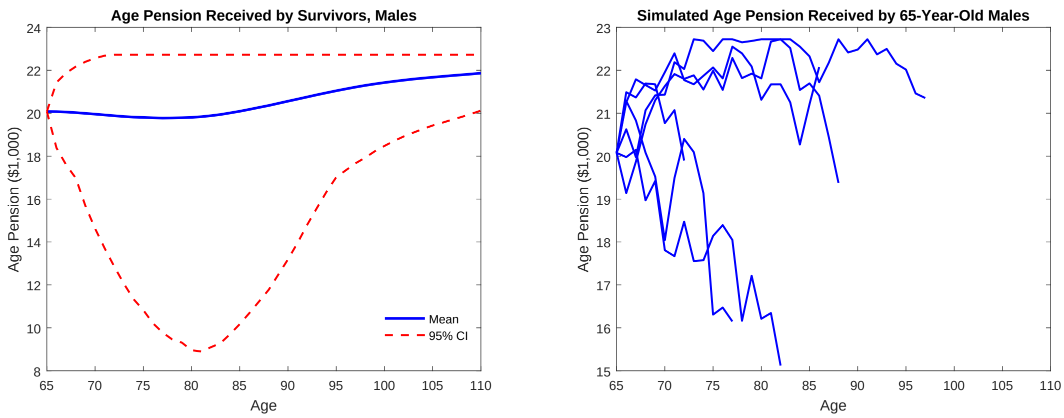


Figure 4.10: Age Pension Entitlements.

4.4.1 Comparison with ASFA Retirement Standard

The Association of Superannuation Funds of Australia (ASFA) provides retirement standards at the level of living a modest lifestyle and of living a comfortable lifestyle, respectively. As of 2015 for 65-year-old singles, the modest lifestyle standard is $23,797 p.a. and the comfortable lifestyle standard is $43,184. [3] We compare our expected consumption levels in the above three cases with ASFA retirement standards. The results are shown in Figure 4.11.

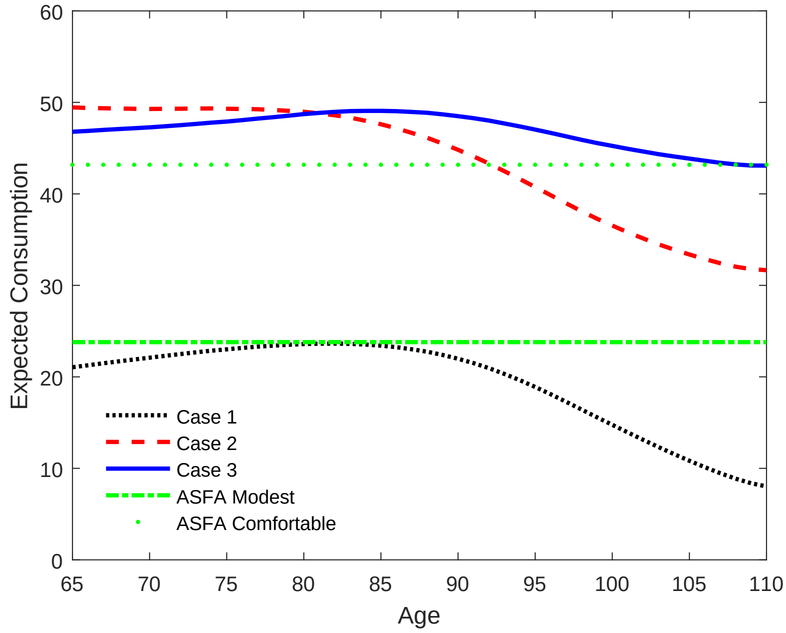


Figure 4.11: Comparison of expected consumption levels for different cases and ASFA retirement standards. In Case 1, retirees have no access to Age Pension or to life annuities; In Case 2, retirees have access to Age Pension; In Case 3, retirees have access to Age Pension as well as life annuities.

As indicated by the blue dotted line in the above figure, for a 65-year-old male with half a million dollars retirement savings, the expected consumption level is below the ASFA modest standard in Case 1, i.e. if the retiree has no access to the Age Pension or commercial life annuities. When the Age Pension is accessible (Case 2), the expected consumption level is substantially higher than the ASFA Modest level and reaches a level around the ASFA comfortable standard, as indicated by the red dashed line. The Age Pension can provide the retiree with a very generous retirement income. Annuitising a proportion of financial assets can result in a generally higher level of consumption. Due to the longevity protection provided by life annuities, the consumption level for older ages in Case 3 (blue solid line) is higher than that in Case 2 (red dashed line).

4.4.2 Asset Allocation Comparison

Allocations to risky assets are compared across the three cases. The results are shown in Figure 4.12. We observe substantially higher allocations to risky assets if the Age Pension is taken into account. This is largely due to the lower-wealth protection provided by the Age Pension.

[3] Data are obtained from ASFA's website via `http://www.superannuation.asn.au/resources/retirement-standard`, on 24 May 2016.

When life annuities are added into the menu the optimal allocation to equities only has slight increase in earlier ages.

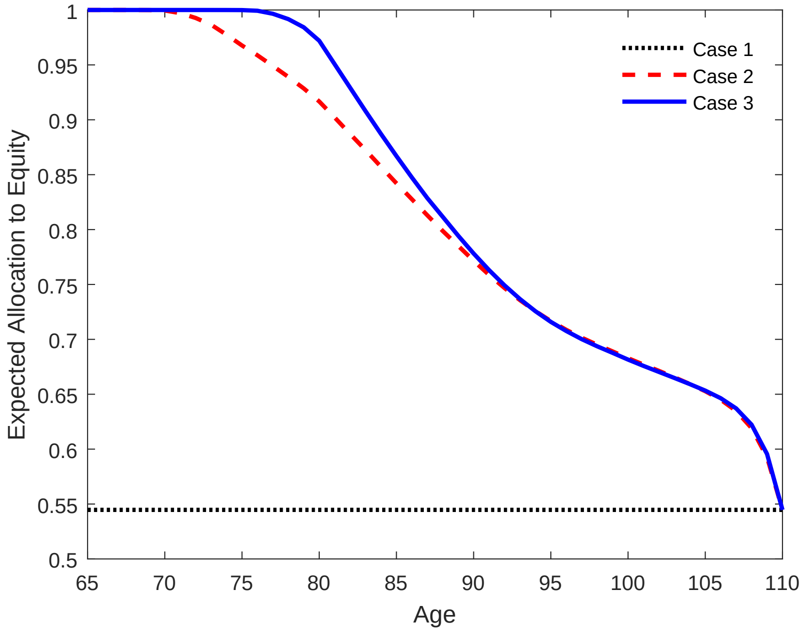


Figure 4.12: Comparison of expected allocations to equity for different cases. In Case 1, retirees have no access to Age Pension or to life annuities; In Case 2, retirees have access to Age Pension; In Case 3, retirees have access to Age Pension as well as life annuities.

4.4.3 Age Pension Entitlements Comparison

The amount of Age Pension entitlements is compared across the three cases. The results are shown in Figure 4.13. We can see the advantages of life annuities in terms of Age Pension entitlements. The advantages are twofold: the average amount of Age Pension is higher for earlier ages (e.g. before age 95 in our case studies); variations are lower.

5 Welfare Analysis

Welfare analysis is widely done to investigate welfare gains of having access to financial products. It can also be used in our case to assess retirees' welfare gains of having the Age Pension provided by the Australian government. This section describes metrics on welfare gains and provides numerical results based on our model set-up.

5.1 Metrics on Welfare Gains

5.1.1 Certainty Equivalent Consumption (CEC)

Certainty Equivalent Consumption is calculated as the consumption level in the one-period utility function, i.e. Equation (2.6), that equates the utility level to the optimal lifetime utility

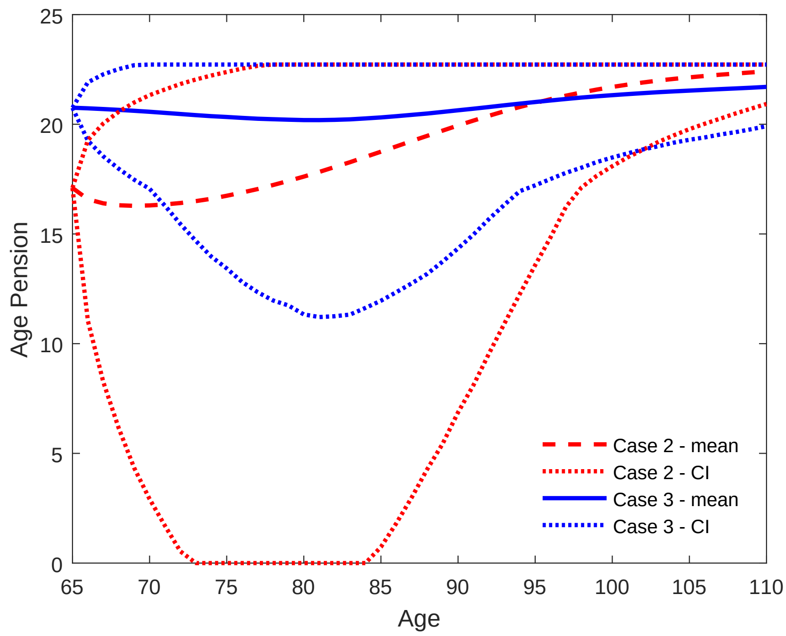


Figure 4.13: Comparison of Age Pension entitlements for different cases. In Case 2, retirees have access to Age Pension only; In Case 3, retirees have access to Age Pension as well as life annuities.

(or value function). CEC, in essence, is a monotonic transformation of the value function. A higher level of value function also corresponds to a higher CEC level. Note that CEC does not necessarily convey information of the actual level of consumption.

5.1.2 Wealth Gap (WG)

Wealth gap is calculated as the additional amount in the initial wealth that can result in the same level of optimal life-time utility as in Case 3. Wealth gap measures the dollar amount of welfare gains, reflected in the initial wealth level, of having access to the Age Pension.

5.1.3 Extra Annual Return (EAR)

Extra annual return is calculated as the additional annual return in the fund investment performance that can result in the same level of optimal life-time utility as in Case 3. Extra annual return measures welfare gains, reflected in terms of investment gains, of having access to the Age Pension.

5.2 Results

Tables 5.1 and 5.2 show the optimal lifetime utility (value function) and the three metrics used for assessing welfare gains of having access to the Age Pension.

We can see a substantial improvement in CEC level when the Age Pension is incorporated. For example, the certainty equivalent consumption in Case 2 increases by 127%. This again confirms that the Australian government is providing a relatively generous Age Pension.

Table 5.1: Value functions and CECs of the 3 cases.

	Case 1	**Case 2**	**Case 3**
Value Function (10^{-5})	-3.05	-0.11	-0.10
CEC ($1,000)	9.52	21.64	22.32

Table 5.2: Welfare gains of having access to the Age Pension (AP) and/or life annuities (LA).

	AP vs Nil	**AP + LA vs AP**	**AP + LA vs Nil**
WG ($1,000)	637.09	39.23	673.01
EAR (%)	8.29	2.52	8.61

The wealth gap of Case 2 vs Case 1 (i.e. Age Pension vs Nil) is $637,000, reflecting the substantial welfare gains of having access to the Age Pension. In other words, the Age Pension provided by the Australian government provides the same welfare gains as if providing a lump sum subsidy of $637,000 at retirement to retirees who have half a million dollars in their account balance. On top of having entitlements to the Age Pension, the provision of commercial life annuities has an extra wealth gap, which is calculated to be $37,000. The wealth gap of Case 3 vs Case 2 (i.e. Age Pension + Life Annuities vs Age Pension) is relatively small compared to that of Age Pension vs Nil. This is because longevity risk is largely reduced by the Age Pension, leaving smaller room for commercial life annuities to take a role.

Comparing EAR in Case 3 and Case 1, we can conclude that the pension fund needs to grow at an extra rate of 8.59% p.a. in order to meet the welfare generated from the Age Pension and life annuities. This extra annual return is indeed very substantial in the current financial environment.

6 Stepping Outside the TUFv1

The life-cycle model proposed in this paper is sophisticated and flexible to be tailored to fit better with specific institutions. After determining a final set of parameter values, we can apply the TUFv1 for better retirement fund design. The TUFv1 is not perfect and there are some aspects that the function can not address and we will need to understand these limitations when we apply the function. As a result, there are cases when a trustee wants to step away from the "straw man".

The reasons could be the trustees:

1. Want to allow for behavioural biases.
2. Have greater member insight.

The trustees that wish to account for these aspects can choose to step outside the TUFv1 through a number of ways:

1. Changing parameter values. Specific examples of altering parameter values are provided in Appendix 9.1. For example, the trustee can turn off bequest motive by setting ϕ equal to 0, as shown in Figure 9.5.
2. Adding in additional constraints to the function, such as consumption floor.

7 Recommendations and Conclusions

In the second part of this paper we have detailed a utility function that we recommend as the Trustee's Utility Function for Default Fund Design v1 (TUFv1). This function aligns with the key starting points outlined in Section 1. We explain the function and provide a detailed example based on account-based pensions.

We recommend that the utility function recommended is adopted as the Trustee's Utility Function for Default Fund Design v1.

The final step that we propose for completing the development side of this project is to produce recommended parameter values for risk aversion (ρ), utility discount factor (β), and bequest motive strength (ϕ).

8 Appendix

8.1 Sensitivity Analysis

We focus on the impact of varying parameters on the allocation to risky asset and on the probability of having negative wealth (called ruin probability thereafter) conditional on survival.

8.1.1 Risk Aversion

We first investigate the impact of the risk aversion parameter (i.e. ρ). Results are shown in Figure 9.1. We observe substantially lower average levels of investment in equity when the trustee assumes more risk-averse members, i.e. the risk aversion parameter is higher. The optimal annuitisation ratio increases as the member becomes more risk-averse, resulting in more risk-averse members to have a flatter expected consumption path and to better utlise the Age Pension but at a cost of generally lower consumption level.

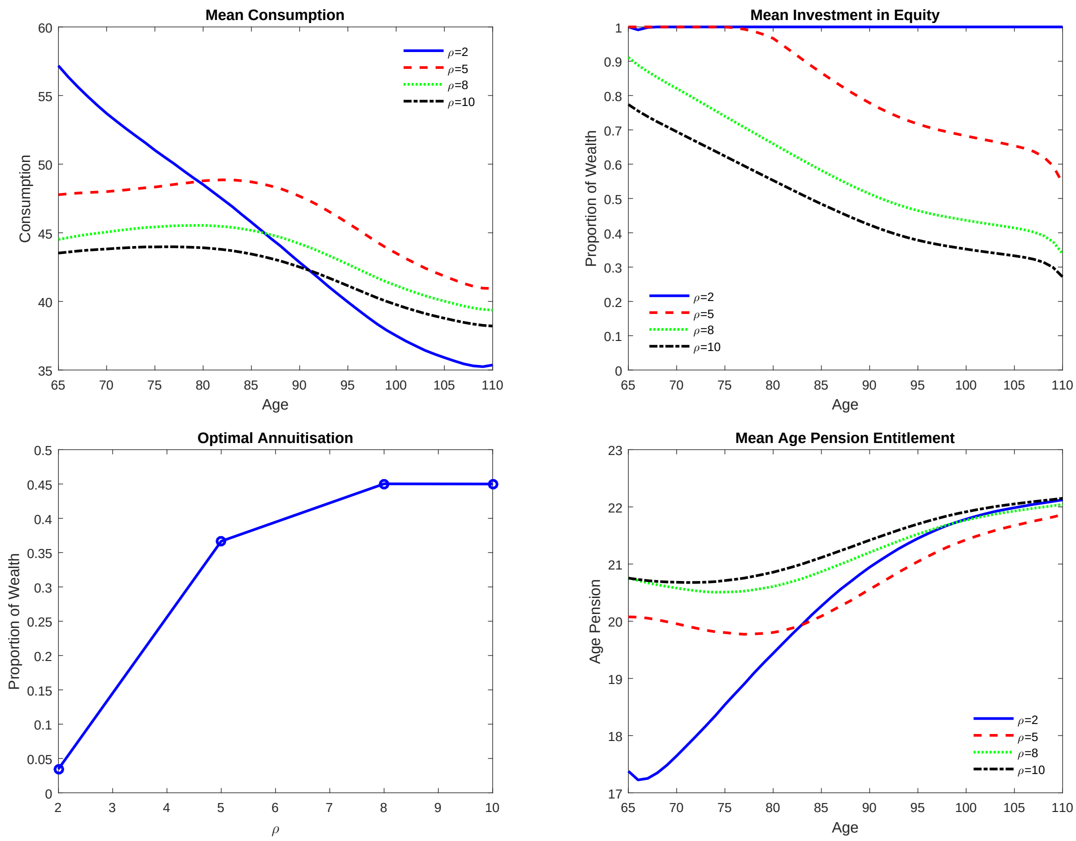


Figure 8.1: Sensitivity analysis of risk aversion parameter on the optimal mean consumption path (top left panel), the optimal mean proportion of wealth invested in equity (top right panel), the optimal annuitisation ratio (bottom left panel), and the mean Age Pension entitlements received (bottom right panel).

8.1.2 Utility Discount Factor

The impact of utility discount factor (i.e. β) is also investigated. Results are shown in Figure 9.2. We can see that the asset allocation and optimal annuitisation are not sensitive to utility discount factors. The optimal annuitisation ratio shows an almost linear relationship with utility discount factor. The more value members put on current vs later consumptions, the optimal annuitisation ratio is lower. The effect of utility discount factor on expected consumption path is indeed the opposite to that of risk-free rate.

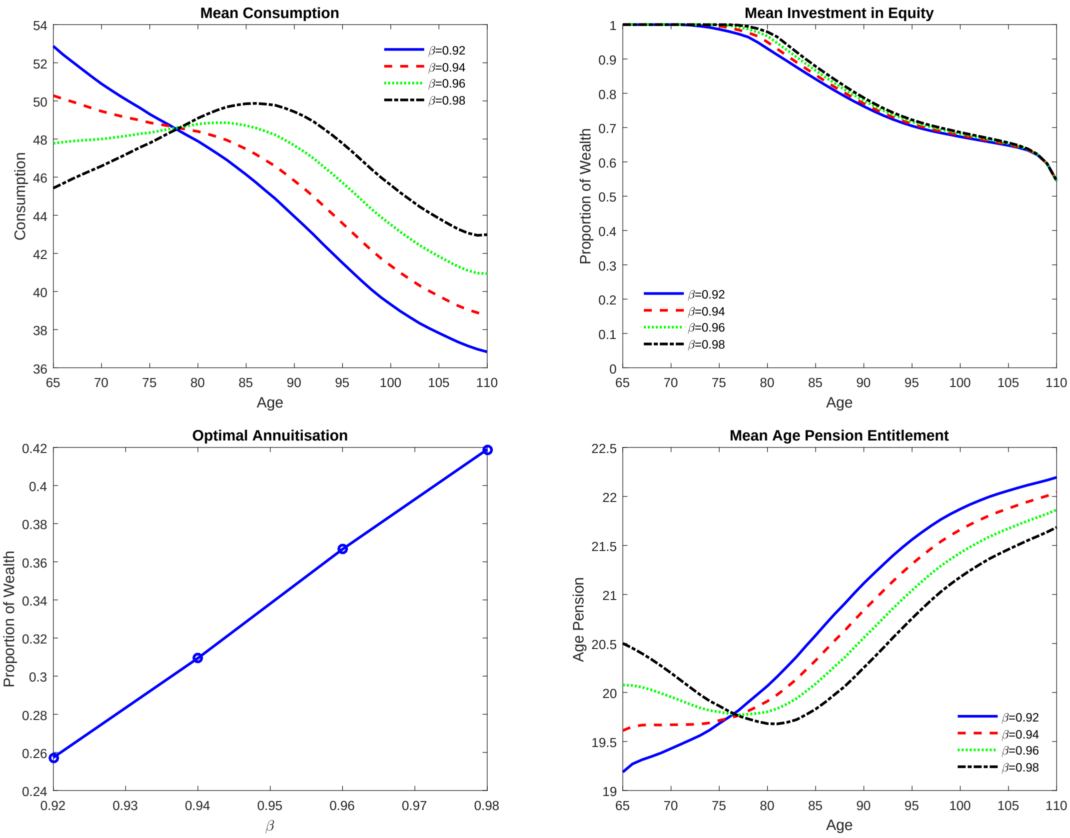


Figure 8.2: Sensitivity analysis of utility discount factor on the optimal mean consumption path (top left panel), the optimal mean proportion of wealth invested in equity (top right panel), the optimal annuitisation ratio (bottom left panel), and the mean Age Pension entitlements received (bottom right panel).

8.1.3 Risk-free Interest Rate

The impact of risk-free interest rate (i.e. r_f) is then investigated. The results are shown in Figure 9.3. We can observe a substantial reduction in equity investment if the risk-free interest rate is more attractive. The value of risk-free interest rate also has a large impact on the optimal annuitisation ratio.

8.1.4 Initial Wealth

Sensitivity test results for the initial wealth (i.e. b_0) are shown in Figure 9.4. The optimal equity investment as a proportion of wealth is not sensitive with respect to initial wealth. The expected consumption level shifts upward as the initial wealth is higher. For retirees with low wealth, the optimal annuitisation ratio is much lower, which is largely due to liquidity constraints. We see that the very wealthy members (i.e. with 2 million dollars initial wealth) can annuitise almost 70% of their asset and have large possibilities of utilising the Age Pension.

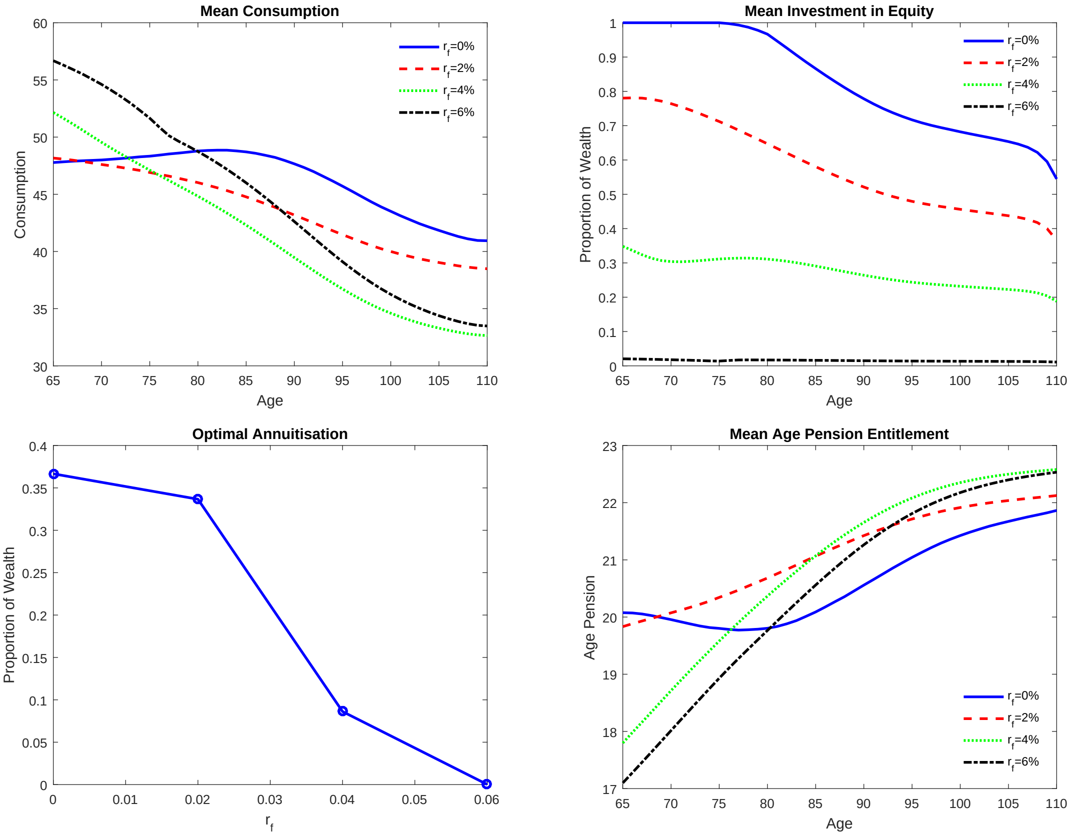


Figure 8.3: Sensitivity analysis of risk-free interest rate on the optimal mean consumption path (top left panel), the optimal mean proportion of wealth invested in equity (top right panel), the optimal annuitisation ratio (bottom left panel), and the mean Age Pension entitlements received (bottom right panel).

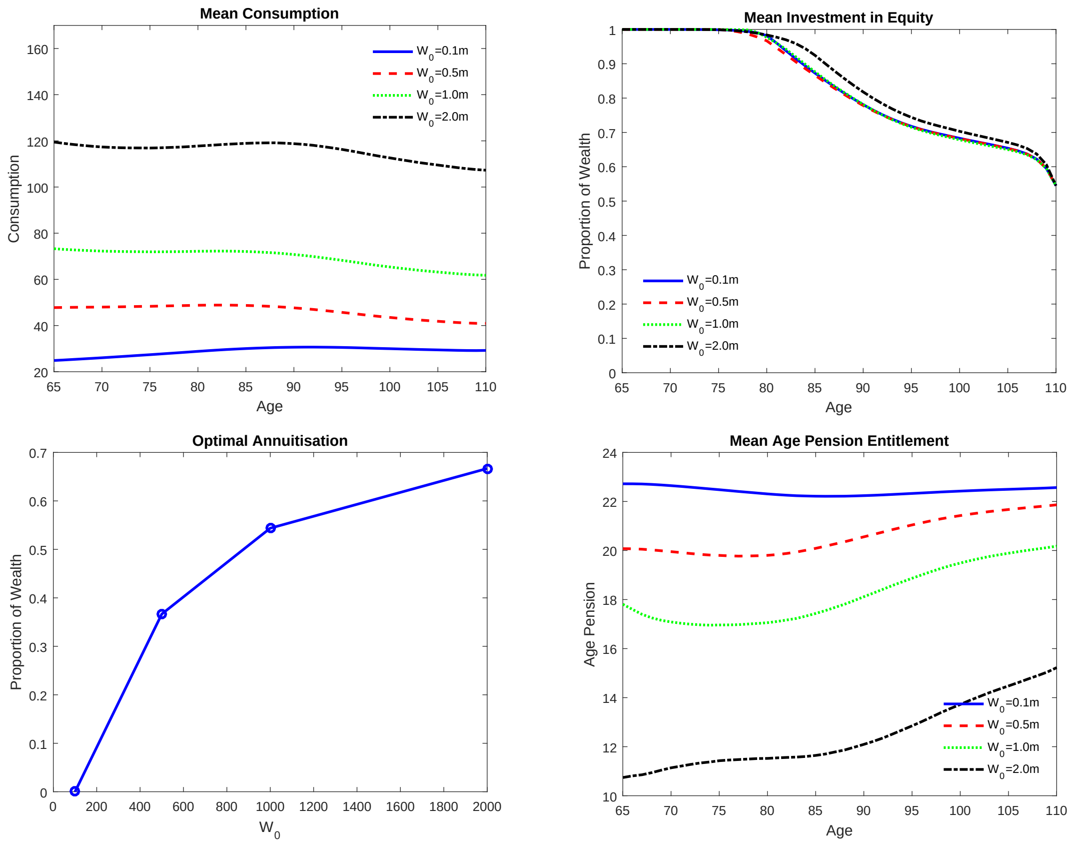


Figure 8.4: Sensitivity analysis of initial wealth on the optimal mean consumption path (top left panel), the optimal mean proportion of wealth invested in equity (top right panel), the optimal annuitisation ratio (bottom left panel), and the mean Age Pension entitlements received (bottom right panel).

8.1.5 Bequest Motive Strength

The impact of bequest motive strength parameter is shown in Figure 9.5, from which we see that a higher bequest motive generally reduces the level of optimal allocation to equity. The optimal annuitisation ratio is largely reduced when the retiree has a very high bequest motive. For members without any bequest motive, i.e. $\phi = 0$, the optimal annuitisation ratio is 0.7 and the optimal expected investment strategy is to invest all assets to equity. The ending point for the blue line in the top right panel (i.e. for $\phi = 0$ in the mean investment strategy plot) becomes 0 or Nil, because the optimal strategy is to consume all assets in the last period for members without any bequest motive. We also observe that the generally decreasing shape of expected consumption with respect to age is averted when the member has a very large bequest motive and thus becomes very thrift on their earlier consumptions. The hump shape in the expected consumption for members without bequest motive is caused by the high annuitisation ratio (i.e. 0.7).

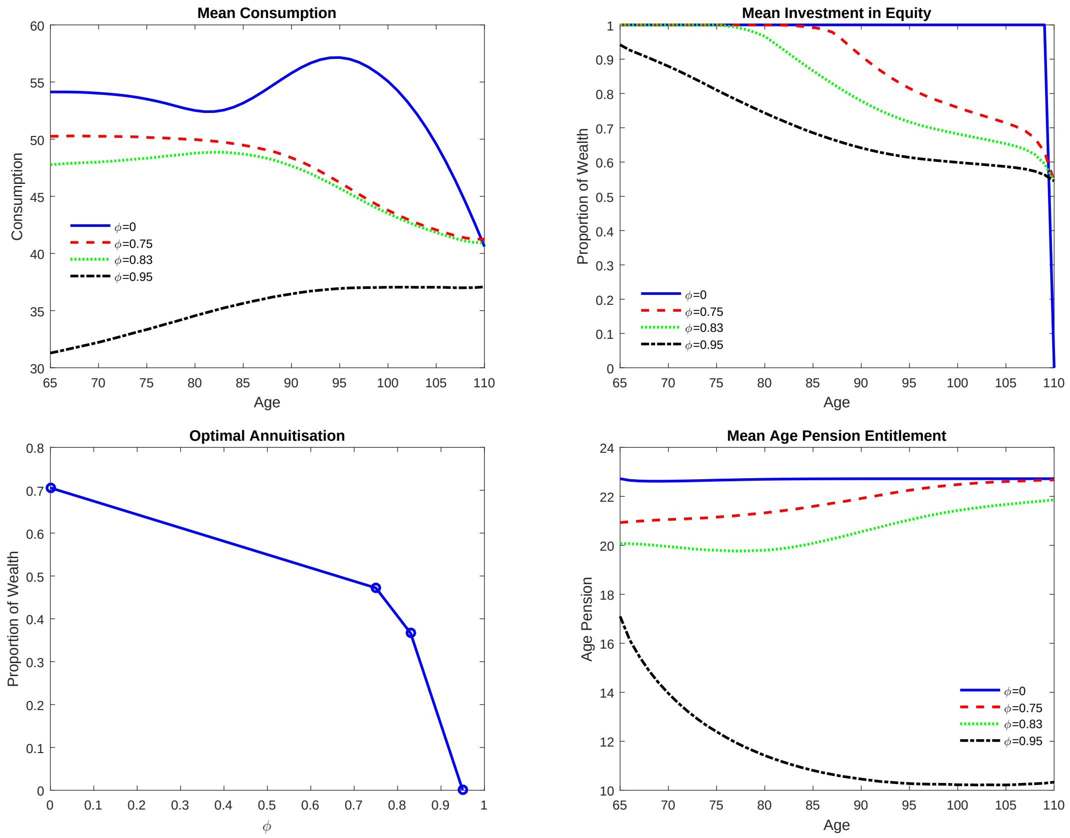


Figure 8.5: Sensitivity analysis of bequest motive strength on the optimal mean consumption path (top left panel), the optimal mean proportion of wealth invested in equity (top right panel), the optimal annuitisation ratio (bottom left panel), and the mean Age Pension entitlements received (bottom right panel).

8.1.6 Equity Return Distributions

We then investigate the sensitivity of our results on equity return distributions. The expected return is first varied from as low as -1% p.a. to as high as 8% p.a. The results are shown in Figure 9.6. The impact of standard deviation of equity returns is shown in Figure 9.7. As expected, the optimal allocation to equity reduces as equity return is sensitive to the trustee's estimation of the distribution of risky assets. The optimal annuitisation increases when equity returns have a lower mean value and a higher volatility, as lower mean and higher volatility both decreases the opportunity cost of annuitisation.

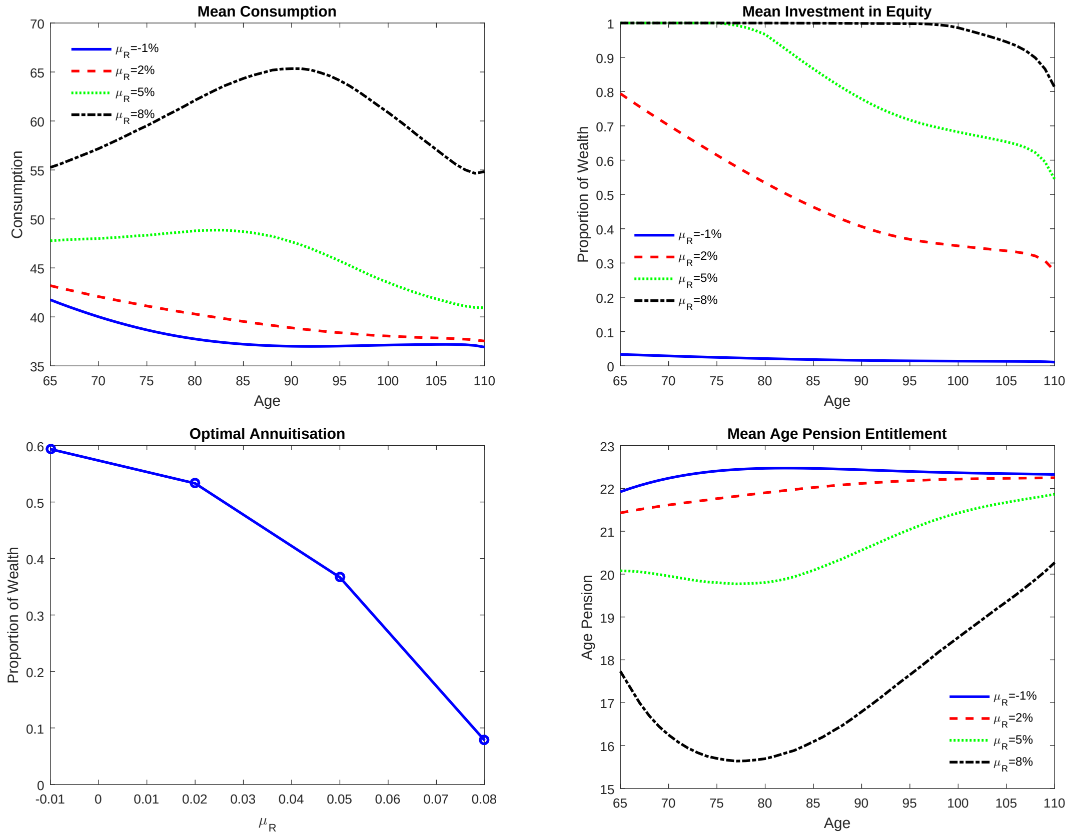


Figure 8.6: Sensitivity analysis of equity return's mean on the optimal mean consumption path (top left panel), the optimal mean proportion of wealth invested in equity (top right panel), the optimal annuitisation ratio (bottom left panel), and the mean Age Pension entitlements received (bottom right panel).

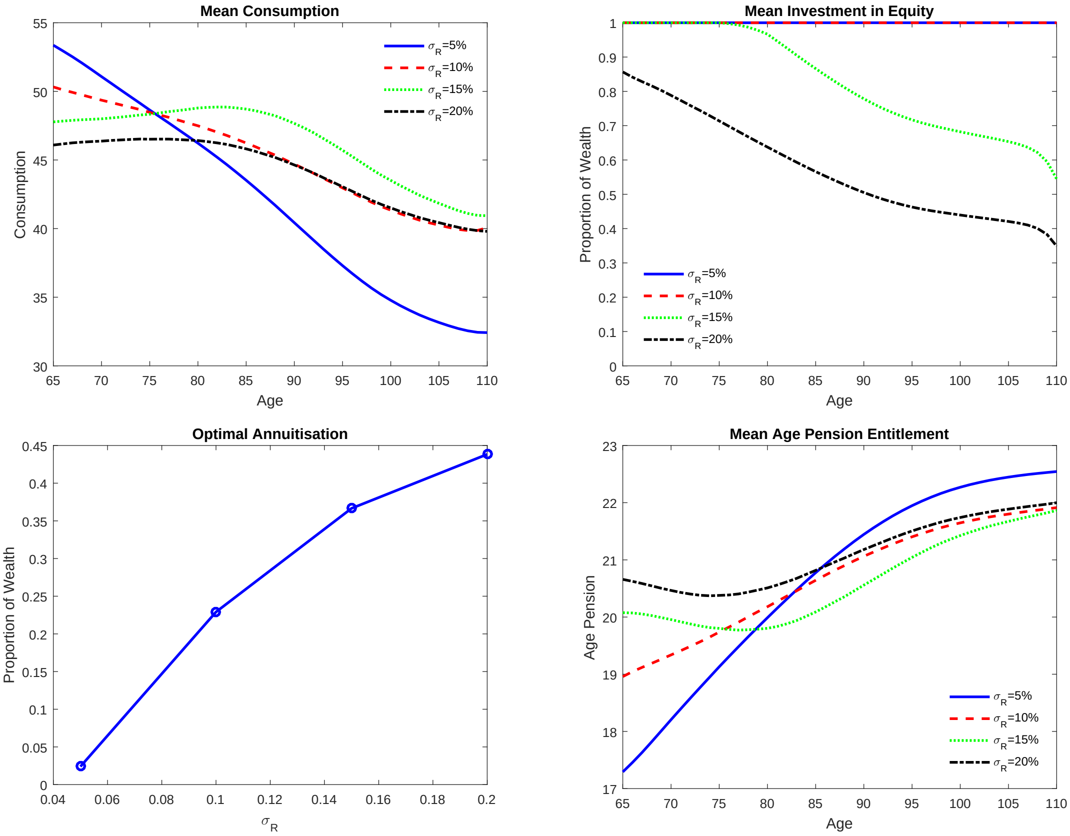


Figure 8.7: Sensitivity analysis of equity return's standard deviation on the optimal mean consumption path (top left panel), the optimal mean proportion of wealth invested in equity (top right panel), the optimal annuitisation ratio (bottom left panel), and the mean Age Pension entitlements received (bottom right panel).

8.2 More Explanation on the Impact of Bequest Motive

8.2.1 A Simple Model

We illustrate the impact of bequest motive on consumption using an N-period simple example where individuals know when they will die and plan to consume a fixed level of consumption throughout their life.

The asset growth rate is assumed to be 0 in this simple example. The utility maximisation objective function can be written as:

$$\begin{aligned} & \max_{\{c\}} \left\{ N\ u(c) + v(b) \right\} \\ = & \max_{\{c\}} \left\{ N \frac{c^{1-\rho}}{1-\rho} + \left(\frac{\phi}{1-\phi} \right)^{\rho} \frac{b^{1-\rho}}{1-\rho} \right\}, \end{aligned} \tag{8.1}$$

where c is the streamlined consumption level and b is the bequest wealth at the pre-determined end of horizon. Given an initial wealth of b_0, the wealth dynamics can be written as

$$b_0 = cN + b. \tag{8.2}$$

The optimal streamlined consumption is obtained by setting the first-order derivative of Equation (9.1) to 0, i.e.

$$\begin{aligned} 0 &= \frac{\partial N\ u(c)}{\partial c} + \frac{\partial v(b)}{\partial c} \\ &= \frac{N}{c^{\rho}} - N \left(\frac{\phi}{1-\phi} \right)^{\rho} \frac{1}{b^{\rho}}. \end{aligned} \tag{8.3}$$

By re-arranging the above equation, we obtain the following relationship between the total consumption vs bequest wealth:

$$\frac{Nc}{b} = \frac{N(1-\phi)}{\phi}. \tag{8.4}$$

In our base case analysis where the bequest motive strength parameter, i.e. ϕ, is equal to 0.83. With a 20 years' horizon, the ratio of total consumption to bequest wealth is 4.10, implying that we are valuing much more on consumption than bequest.

Below is a figure that shows how the ratio of total consumption to bequest wealth changes with respect to different bequest motive strength levels (ϕ) and horizon lengths (N). We see that the ratio of total consumption to bequest wealth increases as the time horizon is longer and the bequest motive strength is lower. For a very extreme case where bequest motive strength of 1, which means that the individual only values bequest, the ratio of total consumption to bequest wealth is constant at 0. For a bequest motive strength of around 0.7, we see that the ratio of total consumption to bequest wealth can be as high as 16 when the time horizon is 30 years. At a bequest motive strength of 0.5, the ratio of total consumption to bequest wealth can be as high as 30 if the individual's horizon is 30 years.

8.2.2 Taking into Account an Approximation of the Age Pension

We use a linear approximation of the Age Pension, i.e. to model means-tested Age Pension entitlement as a linear function with respect to asset level. The calculation of the approximated

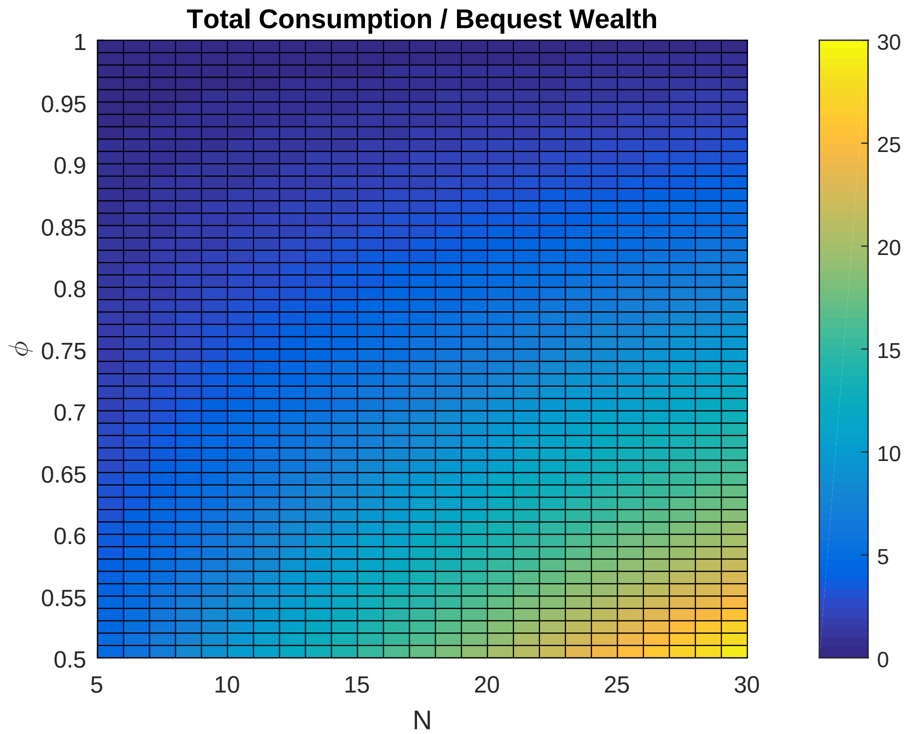


Figure 8.8: Ratio of total consumption to bequest wealth for different bequest motive and horizon lengths, without taking into account the Age Pension.

Age Pension entitlement is:

$$AP_t = \alpha - \tau b_t, \tag{8.5}$$

where AP_t is the approximated Age Pension entitlement for year t, α is the full Age Pension rate, and τ is the approximated Age Pension taper rate that in our case ensures a non-negative approximated Age Pension entitlement.

In the simple framework in this Appendix, the wealth dynamics is as follows:

$$b_t = b_0 - tc + \sum_{k=0}^{t-1} AP_k. \tag{8.6}$$

Starting from the first period, the amount of approximated Age Pension entitlement in this simple model is:

$$\begin{aligned} AP_0 &= \alpha - \tau b_0 \\ AP_1 &= \alpha - \tau\Big(b_0 - c + AP_0\Big) \\ AP_2 &= \alpha - \tau\Big(b_0 - 2c + (AP_0 + AP_1)\Big) \\ &\cdots \\ AP_t &= \alpha - \tau\Big(b_0 - tc + \sum_{k=0}^{t-1} AP_k\Big). \end{aligned} \tag{8.7}$$

The amount of bequest wealth at the end of life in this framework is

$$b = b_0 - Nc + \sum_{k=0}^{N-1} AP_k. \tag{8.8}$$

The partial derivative of the approximated Age Pension entitlement for each period with respect to consumption level is:

$$\begin{aligned}
\frac{\partial}{\partial c} AP_0 &= 0 \\
\frac{\partial}{\partial c} AP_1 &= -\tau\Big(-1 + \frac{\partial}{\partial c} AP_0 \Big) \;=\; \tau \\
\frac{\partial}{\partial c} AP_2 &= -\tau\Big(-2 + \frac{\partial}{\partial c} (AP_0 + AP_1) \Big) \;=\; \tau\,(2-\tau) \\
&\cdots \\
\frac{\partial}{\partial c} AP_t &= -\tau\Big(-t + \sum_{k=0}^{t-1} \frac{\partial}{\partial c} AP_k \Big) \;=\; 1 - \Big(1-\tau\Big)^t .
\end{aligned} \tag{8.9}$$

Therefore, the partial derivative of bequest wealth with respect to consumption is:

$$\begin{aligned}
\frac{\partial b}{\partial c} &= -N + \sum_{k=0}^{N-1} \frac{\partial}{\partial c} AP_k \\
&= -N + \sum_{k=0}^{N-1} \left[1 - \Big(1-\tau\Big)^k \right] \\
&= -\sum_{k=0}^{N-1} \left[\Big(1-\tau\Big)^k \right] \\
&= \begin{cases} -\frac{1-(1-\tau)^N}{\tau} & \tau \neq 0 \\ -N & \tau = 0. \end{cases}
\end{aligned} \tag{8.10}$$

The optimal streamlined consumption is then obtained via the following equation:

$$\begin{aligned}
0 &= \frac{\partial N\ u(c)}{\partial c} + \frac{\partial v(b)}{\partial c} \\
&= \frac{N}{c^\rho} + \frac{\partial b}{\partial c} \left(\frac{\phi}{1-\phi} \right)^\rho \frac{1}{b^\rho} \\
&= \begin{cases} \frac{N}{c^\rho} - \frac{1-(1-\tau)^N}{\tau} \left(\frac{\phi}{1-\phi} \right)^\rho \frac{1}{b^\rho} & \tau \neq 0 \\ \frac{N}{c^\rho} - N \left(\frac{\phi}{1-\phi} \right)^\rho \frac{1}{b^\rho} & \tau = 0. \end{cases}
\end{aligned} \tag{8.11}$$

Solving the above equation, we obtain the optimal ratio of total consumption to bequest wealth as follows:

$$\frac{Nc}{b} = \frac{N(1-\phi)}{\phi} \left(\frac{\tau N}{1-(1-\tau)^N} \right)^{1/\rho} . \tag{8.12}$$

Based on the above results, we see that the ratio of total consumption to bequest wealth increases as the approximated Age Pension taper rate increases and has got nothing to do with the full Age Pension rate. Figure 9.9 shows how the ratio of total consumption to bequest wealth changes with respect to the taper ratio and bequest motive strength where the time horizon is fixed at 20 years.

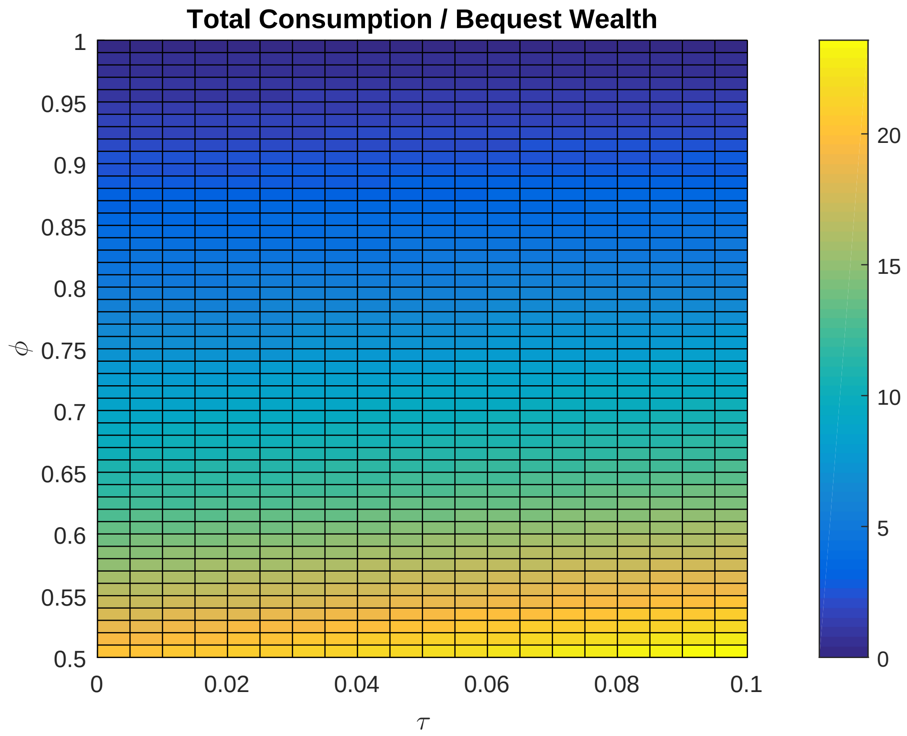


Figure 8.9: Ratio of total consumption to bequest wealth for different bequest motive and approximated Age Pension taper rate, assuming a fixed 20 years' horizon and a risk aversion level of 5 (i.e. $\rho = 5$).

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