Growth curves

There are many growth curves routinely used in the analysis of growth processes that ultimately reach a steady state. These generally form a class of s-shaped or sigmoid curves. These are very useful for modelling populations, labour participation rates, inflation, productivity *growth* (not levels) or other processes where, in the long run, it is expected that the variable will not grow any further. For example, it is not plausible that age-specific per capita use of pharmaceuticals can continue to outstrip GDP growth over the very long run. It might therefore be supposed that in the long run, pharmaceuticals per capita grows at a fixed rate equivalent to GDP growth.

Two growth curves of this kind were used in the analysis in this report (principally in modelling participation and other labour market variables in chapter 3), all of them modified by the inclusion of an additive constant.

The logistic curve

The logistic curve is:

$$y_t = c + \frac{a}{(1 + b\exp(gt))}$$

At t=0, y(0) = c+a/(1+b)

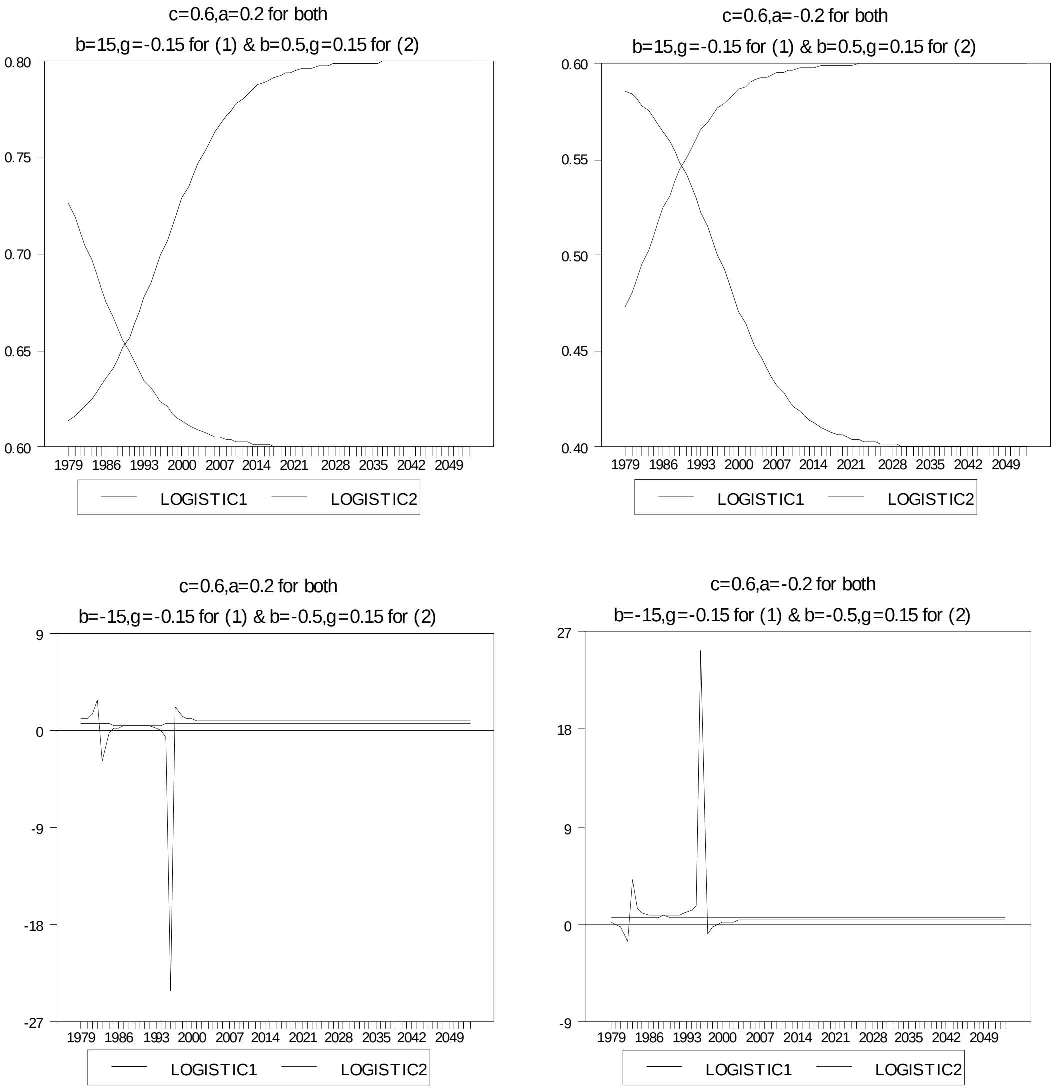
At t=∞, then if g<0, y(∞) = c+a , else if g>0, y(∞) = c.

Where g<0, a>0 and b>0, then the logistic is a positively sloped growth curve that reaches the saturation point (c+a) from below (figure 2.1). Where g, a and b>0 then the logistic is negatively sloped and asymptotes to c from above.

Where g<0, a<0 and b>0, then the logistic is negatively sloped and asymptotes to c+a from above. Where g>0, a<0 and b>0 then the logistic is positively sloped and reaches a saturation point of c from below.

Where b<0 the logistic exhibits irregularities in its growth that make it unsuited to most growth processes (figure 2.1). It is probably sensible to impose the condition that b>0 in most empirical applications.

Figure 2.1 **Logistic curves**



Getting reliable estimates of the parameters of a logistic requires some knowledge of the time of inflection in the curve (i.e. the time at which the absolute value of the growth rate is maximised). The inflection point of a logistic is when:

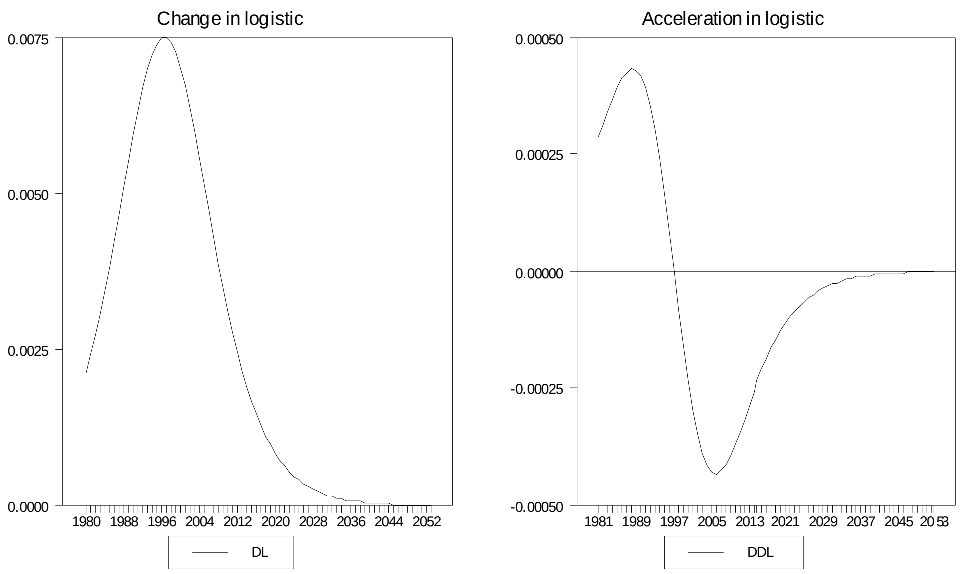
$$t = -\frac{1}{g}\ln(b)$$

which is only defined when b>0. The inflection point of a logistic is relatively inflexible. In all cases, the remaining growth in y from the inflection point is fixed at a/2.

In many cases, the observed data do not show an obvious inflection point. Estimating a logistic function on such data without imposing an assumption about when the inflection will occur will often give nonsensical results. This is true for all other s-shaped functions. However, sometimes it may be possible to infer inflection points based on prior knowledge or extrapolation of the double difference in the data series. It is apparent from figures 2.2 and 2.3 that the inflection point is where the double differenced data crosses the time axis. So even if an inflection point is not observed, one strategy is to:

- smooth the data to eliminate high frequency cycles in the data (such as through the use of a Hodrick-Prescott filter); and
- double difference the smoothed data and guess at what time it will reach zero. This 'flex' point can then be imposed in any estimation of the function. If the inflection point, f, can be calculated in this way then it implies that $b = e^{-gf}$, which reduces the parameters to be estimated.

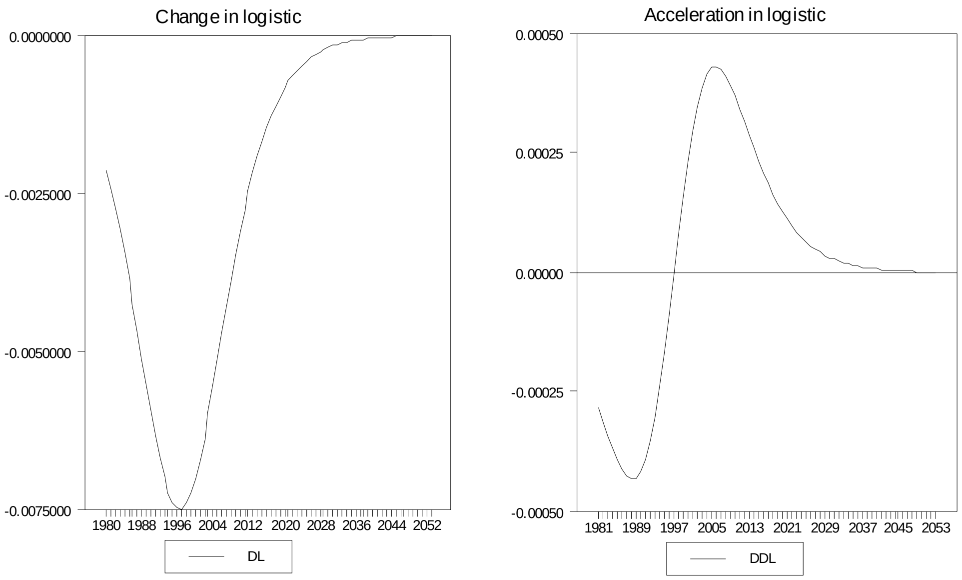
Figure 2.2 **Growth and acceleration of logistic curves**[a]
Positively sloped logistic



a Based on the parameters of LOGISTIC 1 shown in the left hand top logistic curve in the previous figure.

Figure 2.3 **Growth and acceleration of logistic curves**[a]

Negatively sloped logistic



[a] Based on the parameters of LOGISTIC 1 shown in the right hand top logistic curve in the previous figure.

The Richards curve

This is a very flexible growth curve denoted by:

$$y_t = c + a \times (1 + be^{gt})^{\lambda}$$

The Richards curve translates into many other growth curves for different values of λ (figure 2.4). It is a logistic where $\lambda = -1$, a Gompertz for $\lambda = \pm\infty$, and a Bertalanffy function at $\lambda = 3$.

At t=0, $y(0) = c + a \times (1 + b)^{\lambda}$

At t=∞, then if g<0, $y(\infty) = c + a$, as with the logistic.

The inflection point of a Richards function, which depends on λ, is not in fixed proportion to its asymptote. The time at inflection is:

$$t = -\frac{1}{g}\ln(-b\lambda)$$

If there is sufficient data, then the Richards curve can be estimated using non-linear least squares. However, there are often problems in convergence, and imprecise estimates are obtained if the data does not already include the inflection point. In order that absurd estimates are not produced with shorter datasets, it is sometimes appropriate to impose restrictions on the estimation.

First, it is often the case that a limit beyond which the curve will not go can be defined (L). In modelling, the long run can be restricted so that it does not exceed this limit. We imposed this condition using the following approach $c + a = y(m) + \{1/(1+e^{\phi})\} \times (L - y(m))$,where y is the value of the observed curve at the last point (t = m). The term, $1/(1+e^{\phi})$, is bounded between 0 and 1, depending on the value estimated by non-linear least squares for ϕ, allowing some latitude in reaching the limit.

Second, there may be prior information that indicates that a particular point is likely to lie on the curve (say when t = v). In that case, $y_v = c + a \times (1 + be^{gv})^{\lambda}$. That, combined with information about the limit above, implies that:

$$a = \frac{y(v) - y(m) - \{1 - 0.5e^{\phi}/(1+e^{\phi})\} \times (L - y(m))}{(1+be^{gv})^{\lambda} - 1}$$

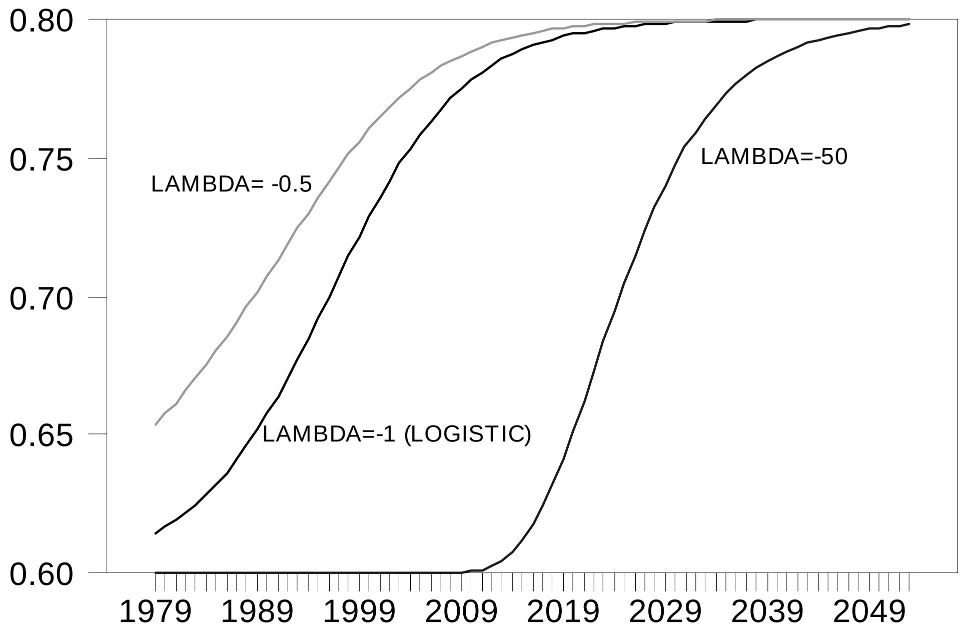
so that *a* may not have to be estimated.

Third, if the inflection point, t = f, can be anticipated, then one further parameter need not be estimated since $b = -\frac{1}{\lambda}e^{(-gf)}$.

In that case, the parameters g, λ and ϕ need only be estimated.

Figure 2.4 **Richards' curves**[a]

At different values of λ



[a] The parameter values are as in LOGISTIC1 in figure Z.1, except that λ varies.